

Finitely based monoids

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Abstract

We present a method for proving that a semigroup is finitely based and find some new sufficient conditions under which a monoid is finitely based. Our method also gives a short proof to the theorem of E. Lee that every monoid that satisfies $xt_1xyt_2y \approx xt_1yxt_2y$ and $xyt_1xt_2y \approx yxt_1xt_2y$ is finitely based.

Keywords: Finite Basis Problem, Semigroups, Monoids

1 Introduction

An algebra is said to be *finitely based* (FB) if there is a finite subset of its identities from which all of its identities may be deduced. Otherwise, an algebra is said to be *non-finitely based* (NFB). Famous Tarski's Finite Basis Problem asks if there is an algorithm to decide when a finite algebra is finitely based. In 1996, R. McKenzie [5] solved this problem in the negative showing that the classes of FB and inherently not finitely based finite algebras are recursively inseparable. (A locally finite algebra is said to be *inherently not finitely based* (INFB) if any locally finite variety containing it is NFB.)

It is still unknown whether the set of FB finite semigroups is recursive although a very large volume of work is devoted to this problem (see the surveys [13, 14]). In contrast with McKenzie's result, a powerful description of the INFB finite semigroups has been obtained by M. Sapir [7, 8]. These results show that we need to concentrate on NFB finite semigroups that are not INFB.

In 1976, M. Sapir suggested to concentrate on the class of monoids of the form $S(W)$. (A *monoid* is a semigroup with an identity element.) Monoids of the form $S(W)$ are defined as follows.

Let \mathfrak{A} be an alphabet and W be a set of words in the free monoid \mathfrak{A}^* . Let $S(W)$ denote the Rees quotient over the ideal of \mathfrak{A}^* consisting of all words that are not subwords of words in W . For each set of words W , the semigroup $S(W)$ is a monoid with zero whose nonzero elements are the subwords of words in W . Evidently, $S(W)$ is finite if and only if W is finite.

The identities of these semigroups have been of interest since P. Perkins [6] showed that $S(\{abtb, atbab, abab, aat\})$ was NFB. It was one of the first examples of a finite NFB semigroup. It is clear from the results of [7, 8] that a semigroup of the form $S(W)$ is never INFB. It is shown in [2] that the class of monoids of the form $S(W)$ is as “bad” with respect to the finite basis property as the class of all finite semigroups. In particular, the set of FB semigroups and the set of NFB semigroups in this class are not closed under taking direct products, and there exists an infinite chain of varieties generated by such semigroups where FB and NFB varieties alternate.

We use $\text{var}\Delta$ to denote the variety defined by a set of identities Δ and $\text{var}S$ to denote the variety generated by a semigroup S . The identities $xt_1xyt_2y \approx xt_1yxt_2y$, $xyt_1xt_2y \approx yxt_1xt_2y$ and $xt_1yt_2xy \approx xt_1yt_2yx$ we denote respectively by σ_μ , σ_1 and σ_2 . Notice that the identities σ_1 and σ_2 are dual to each other.

In [1], M. Jackson proved that $\text{var}S(\{at_1abt_2b\})$ and $\text{var}S(\{abt_1at_2b, at_1bt_2ab\})$ are *limit varieties* in a sense that each of these varieties is NFB while each proper monoid subvariety of each of these varieties is FB. In order to determine whether $\text{var}S(\{at_1abt_2b\})$ and $\text{var}S(\{abt_1at_2b, at_1bt_2ab\})$ are the only limit varieties generated by finite aperiodic monoids with central idempotents, he suggested in [1] to investigate the monoid subvarieties of $\text{var}\{\sigma_\mu, \sigma_1\}$ and dually, of $\text{var}\{\sigma_\mu, \sigma_2\}$. In [3], E. Lee proved that all finite aperiodic monoids with central idempotents contained in $\text{var}\{\sigma_\mu, \sigma_1\}$ are finitely based. This result implies the affirmative answer to the question of Jackson posed in [1]. Later in [4], E. Lee proved that all monoids contained in $\text{var}\{\sigma_\mu, \sigma_1\}$ are finitely based. This more general result implies that $\text{var}S(\{at_1abt_2b\})$ and $\text{var}S(\{abt_1at_2b, at_1bt_2ab\})$ are the only limit varieties generated by aperiodic monoids with central idempotents.

In this article we present a method (see Lemma 3.2 below) that can be used for proving that a semigroup is finitely based. By using this method we give a short proof to the result of Lee that every monoid contained in $\text{var}\{\sigma_\mu, \sigma_1\}$ is finitely based (see Theorem 3.5 below). We also use our method to find some new sufficient conditions under which a monoid is finitely based. These results will be used in articles [11, 12].

If a variable t occurs exactly once in a word \mathbf{u} then we say that t is *linear* in \mathbf{u} . If a variable x occurs more than once in a word \mathbf{u} then we say that x is *non-linear* in \mathbf{u} . Articles [11, 12] contain some algorithms that recognize FB semigroups among certain finite monoids of the form $S(W)$. In particular, in article [11], we show how to recognize FB semigroups among the monoids of the form $S(W)$ where W consists of a single word with at most two non-linear variables. It follows from [11] that if W consists of a single word with at most two non-linear variables and the monoid $S(W)$ is finitely based then $S(W)$ is contained either in $\text{var}\{\sigma_\mu, \sigma_1\}$ or in $\text{var}\{\sigma_\mu, \sigma_2\}$ or in $\text{var}\{\sigma_1, \sigma_2\}$. As another application of our method, we give a simple description of the equational theories for each of the seven monoid varieties defined by the subsets of $\{\sigma_\mu, \sigma_1, \sigma_2\}$. We also show that the monoid varieties defined by $\{\sigma_\mu, \sigma_1, \sigma_2\}$ and by $\{\sigma_1, \sigma_2\}$ are generated by monoids of the form $S(W)$.

2 Preliminaries

Throughout this article, elements of a countable alphabet \mathfrak{A} are called *variables* and elements of the free semigroup \mathfrak{A}^+ are called *words*. If some variable x occurs $n \geq 0$ times in a word \mathbf{u} then we write $\text{occ}_{\mathbf{u}}(x) = n$ and say that x is *n-occurring* in \mathbf{u} . The set $\text{Cont}(\mathbf{u}) = \{x \in \mathfrak{A} \mid \text{occ}_{\mathbf{u}}(x) > 0\}$ of all variables contained in a word \mathbf{u} is called the *content* of \mathbf{u} . For each $n > 0$ we define $\text{Cont}_n(\mathbf{u}) = \{x \in \mathfrak{A} \mid 0 < \text{occ}_{\mathbf{u}}(x) \leq n\}$. We use $\text{Lin}(\mathbf{u})$ to denote the set $\text{Cont}_1(\mathbf{u})$ of all linear variables in \mathbf{u} . If \mathfrak{X} is a set of variables then we write $\mathbf{u}(\mathfrak{X})$ to refer to the word obtained from \mathbf{u} by deleting all occurrences of all variables that are not in \mathfrak{X} and say that the word \mathbf{u} *deletes* to the word $\mathbf{u}(\mathfrak{X})$. If $\mathfrak{X} = \{y_1, \dots, y_k\} \cup \mathfrak{Y}$ for some variables y_1, \dots, y_k and a set of variables \mathfrak{Y} then instead of $\mathbf{u}(\{y_1, \dots, y_k\} \cup \mathfrak{Y})$ we simply write $\mathbf{u}(y_1, \dots, y_k, \mathfrak{Y})$.

We say that a set of identities Σ is closed under deleting variables, if for each set of variables \mathfrak{X} , the set Σ contains the identity $\mathbf{u}(\mathfrak{X}) \approx \mathbf{v}(\mathfrak{X})$ whenever Σ contains an identity $\mathbf{u} \approx \mathbf{v}$. We use Σ^δ to denote the closure of Σ under deleting variables. For example, $\{\sigma_\mu\}^\delta = \{xt_1xyt_2y \approx xt_1yxt_2y, xxyt_2y \approx xyxt_2y, xt_1xyy \approx xt_1yxy, xxyy \approx xyxy\}$. If a semigroup S satisfies all identities in a set Σ then we write $S \models \Sigma$. If S is a monoid then evidently, $S \models \Sigma$ if and only if $S \models \Sigma^\delta$.

A *block* of a word \mathbf{u} is a maximal subword of \mathbf{u} that does not contain any linear letters of \mathbf{u} . An identity $\mathbf{u} \approx \mathbf{v}$ is called *regular* if $\text{Cont}(\mathbf{u}) = \text{Cont}(\mathbf{v})$. An identity $\mathbf{u} \approx \mathbf{v}$ is called *balanced* if for each variable $x \in \mathfrak{A}$ we have $\mathbf{u}(x) = \mathbf{v}(x)$. For each $n > 0$ an identity $\mathbf{u} \approx \mathbf{v}$ is called a \mathcal{P}_n -*identity* if it is regular and $\mathbf{u}(\text{Cont}_n(\mathbf{u})) = \mathbf{v}(\text{Cont}_n(\mathbf{u}))$. In particular, an identity is a \mathcal{P}_1 -identity if and only if it is regular and the order of linear letters is the same in both of its sides. An identity $\mathbf{u} \approx \mathbf{v}$ is called *block-balanced* if for each variable $x \in \mathfrak{A}$, we have $\mathbf{u}(x, \text{Lin}(\mathbf{u})) = \mathbf{v}(x, \text{Lin}(\mathbf{u}))$. Evidently, an identity $\mathbf{u} \approx \mathbf{v}$ is block-balanced if and only if it is a balanced \mathcal{P}_1 -identity and each block in \mathbf{u} is a permutation of the corresponding block in \mathbf{v} .

A word that contains at most one non-linear variable is called *almost-linear*. An identity $\mathbf{u} \approx \mathbf{v}$ is called *almost-linear* if both words \mathbf{u} and \mathbf{v} are almost-linear.

Fact 2.1. *If the word xy is not an isoterm for a monoid S and $S \models \sigma_\mu$ then S is either finitely based by some almost-linear identities or $S \models x \approx x^n$ for some $n > 1$ and satisfies only regular identities.*

Proof. If S satisfies an irregular identity then S is a group with period $n > 0$. Since S satisfies the identity $xyy \approx xyxy$, the group S is finitely based by $\{y \approx x^n y \approx x^n y, xy \approx yx\}$. So, we may assume that S satisfies only regular identities.

Since the word xy is not an isoterm for S , the monoid S satisfies a non-trivial identity of the form $xy \approx \mathbf{u}$. Since S satisfies only regular identities, we have that $\text{Cont}(\mathbf{u}) = \{x, y\}$. If the length of the word \mathbf{u} is 2 then S is commutative and is finitely based by either $\{x^m \approx x, xy \approx yx\}$ for some $m > 1$ or by $xy \approx yx$. If the length of the word \mathbf{u} is at least 3 then S satisfies an identity $x \approx x^n$ for some $n > 1$. \square

We say that a set of identities Σ is *finitely based* if all identities in Σ can be derived from a finite subset of identities in Σ .

Lemma 2.2. [15, Corollary 2] *Every set of almost-linear identities is finitely based.*

Lemma 2.3. [4, Proposition 5.7] *Every set of identities that consists of $\{\sigma_\mu, \sigma_1\}^\delta$ and some identities of the form*

$$x^{\alpha_0} y^{\beta_0} t_1 x^{\alpha_1} y^{\beta_1} t_2 \dots t_n x^{\alpha_n} y^{\beta_n} \approx y^{\beta_0} x^{\alpha_0} t_1 x^{\alpha_1} y^{\beta_1} t_2 \dots t_n x^{\alpha_n} y^{\beta_n} \quad (1)$$

where $\alpha_0, \beta_0 > 0$ and $n, \alpha_1, \beta_1, \dots, \alpha_n, \beta_n \geq 0$, is finitely based.

The cardinality of a set X is denoted by $|X|$. We refer the reader to the article [10] for some examples that illustrate the definitions given below.

We use $_{i\mathbf{u}}x$ to refer to the i^{th} from the left occurrence of x in \mathbf{u} . We use $_{\ell\mathbf{u}}x$ to refer to the last occurrence of x in \mathbf{u} . The set $\text{OccSet}(\mathbf{u}) = \{_{i\mathbf{u}}x \mid x \in \mathfrak{A}, 1 \leq i \leq \text{occ}_{\mathbf{u}}(x)\}$ of all occurrences of all variables in \mathbf{u} is called the *occurrence set of \mathbf{u}* . The word \mathbf{u} induces a (total) order $<_{\mathbf{u}}$ on set $\text{OccSet}(\mathbf{u})$ defined by $_{i\mathbf{u}}x <_{\mathbf{u}} _{j\mathbf{u}}y$ if and only if the i^{th} occurrence of x precedes the j^{th} occurrence of y in \mathbf{u} .

If \mathbf{u} and \mathbf{v} are two words then $l_{\mathbf{u},\mathbf{v}}$ is a map from $\{_{i\mathbf{u}}x \mid x \in \text{Cont}(\mathbf{u}), i \leq \min(\text{occ}_{\mathbf{u}}(x), \text{occ}_{\mathbf{v}}(x))\}$ to $\{_{i\mathbf{v}}x \mid x \in \text{Cont}(\mathbf{v}), i \leq \min(\text{occ}_{\mathbf{u}}(x), \text{occ}_{\mathbf{v}}(x))\}$ defined by $l_{\mathbf{u},\mathbf{v}}(_{i\mathbf{u}}x) = _{i\mathbf{v}}x$. If $X \subseteq \text{OccSet}(\mathbf{u})$ then we say that set X is *left-stable* in an identity $\mathbf{u} \approx \mathbf{v}$ if the map $l_{\mathbf{u},\mathbf{v}}$ is defined on X and is an isomorphism of the (totally) ordered sets $(X, <_{\mathbf{u}})$ and $(l_{\mathbf{u},\mathbf{v}}(X), <_{\mathbf{v}})$. Otherwise, we say that set X is *left-unstable* in $\mathbf{u} \approx \mathbf{v}$. If $\mathbf{u} \approx \mathbf{v}$ is a balanced identity then the map $l_{\mathbf{u},\mathbf{v}}$ is defined on every $X \subseteq \text{OccSet}(\mathbf{u})$. In this case, instead of saying that X is left-stable (or left-unstable) in $\mathbf{u} \approx \mathbf{v}$ we simply say that X is *stable* or (*unstable*) in $\mathbf{u} \approx \mathbf{v}$.

3 A method for proving that a semigroup is finitely based

If a pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ is adjacent in \mathbf{u} and $c <_{\mathbf{u}} d$ then we write $c \ll_{\mathbf{u}} d$.

Fact 3.1. [9, Lemma 3.2] *If $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ is left-unstable in an identity $\mathbf{u} \approx \mathbf{v}$ and $c <_{\mathbf{u}} d$ then for some $\{p, q\} \subseteq \text{OccSet}(\mathbf{u})$ we have that $c \leq_{\mathbf{u}} p \ll_{\mathbf{u}} q \leq_{\mathbf{u}} d$ and $\{p, q\}$ is also left-unstable in $\mathbf{u} \approx \mathbf{v}$.*

If $\mathbf{u} \approx \mathbf{v}$ is a balanced identity then for each $x \in \mathfrak{A}$ and $1 \leq i \leq \text{occ}_{\mathbf{u}}(x) = \text{occ}_{\mathbf{v}}(x)$ we identify $_{i\mathbf{u}}x \in \text{OccSet}(\mathbf{u})$ and $_{i\mathbf{v}}x \in \text{OccSet}(\mathbf{v})$. We say that a pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ is *critical* in a balanced identity $\mathbf{u} \approx \mathbf{v}$ if $\{c, d\}$ is adjacent in \mathbf{u} and unstable in $\mathbf{u} \approx \mathbf{v}$. Fact 3.1 implies that every non-trivial balanced identity $\mathbf{u} \approx \mathbf{v}$ contains a critical pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$.

An *assignment of Types* is a function that assigns values (Types) from 1 to n to every pair of occurrences of distinct variables in all words. Each assignment of Types induces a function on balanced identities. We say that a balanced identity $\mathbf{u} \approx \mathbf{v}$ is of *Type k* if k is the maximal number so that the identity $\mathbf{u} \approx \mathbf{v}$ contains an unstable pair of Type k . If $\mathbf{u} \approx \mathbf{v}$ does not contain any unstable pairs (i.e. trivial) then we say that $\mathbf{u} \approx \mathbf{v}$ is of Type 0.

We say that a property \mathcal{P} of identities is *transitive* if an identity $\mathbf{u} \approx \mathbf{v}$ satisfies \mathcal{P} whenever both $\mathbf{u} \approx \mathbf{w}$ and $\mathbf{w} \approx \mathbf{v}$ satisfy \mathcal{P} . It is easy to see that all properties of identities that we defined in Section 2 are transitive. The following lemma can be used to prove that a semigroup is finitely based.

Lemma 3.2. *Let \mathcal{P} be a transitive property of identities which is at least as strong as the property of being a balanced identity. Let Δ be a set of \mathcal{P} -identities. Suppose that one can find an assignment of Types from 1 to n so that for each $1 \leq i \leq n$, if a \mathcal{P} -identity $\mathbf{u} \approx \mathbf{v}$ contains a critical pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of Type i then one can derive a \mathcal{P} -identity $\mathbf{u} \approx \mathbf{w}$ from Δ so that*

- (i) *the pair $\{c, d\}$ is stable in $\mathbf{w} \approx \mathbf{v}$;*
 - (ii) *each pair of Type $\geq i$ is stable in $\mathbf{w} \approx \mathbf{v}$ whenever it is stable in $\mathbf{u} \approx \mathbf{v}$.*
- Then every \mathcal{P} -identity can be derived from Δ .*

Proof. For each $1 \leq i \leq n$, we use $\text{Chaos}_i(\mathbf{x} \approx \mathbf{y})$ to denote the set of all unstable pairs of Type i in a balanced identity $\mathbf{x} \approx \mathbf{y}$.

Claim 1. *Let $\mathbf{u} \approx \mathbf{v}$ be a \mathcal{P} -identity of Type k for some $1 \leq k \leq n$. Then one can derive a \mathcal{P} -identity $\mathbf{u} \approx \mathbf{w}$ from Δ so that the identity $\mathbf{w} \approx \mathbf{v}$ is also of Type k and contains a critical pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{w})$ of Type k .*

Proof. Since $\mathbf{u} \approx \mathbf{v}$ is of Type k , it contains an unstable pair of Type k . Then by Fact 3.1, the identity $\mathbf{u} \approx \mathbf{v}$ contains a critical pair $\{a_1, b_1\} \subseteq \text{OccSet}(\mathbf{u})$. The pair $\{a_1, b_1\}$ is of Type $T_1 \in \{1, 2, \dots, k\}$. By our assumption, one can derive a \mathcal{P} -identity $\mathbf{u} \approx \mathbf{p}_1$ from Δ so that for each $i > T_1$ we have $\text{Chaos}_i(\mathbf{p}_1 \approx \mathbf{v}) = \text{Chaos}_i(\mathbf{u} \approx \mathbf{v})$ and $\text{Chaos}_{T_1}(\mathbf{p}_1 \approx \mathbf{v})$ is a proper subset of $\text{Chaos}_{T_1}(\mathbf{u} \approx \mathbf{v})$.

If the identity $\mathbf{p}_1 \approx \mathbf{v}$ is non-trivial, then by Fact 3.1, it contains a critical pair $\{a_2, b_2\} \subseteq \text{OccSet}(\mathbf{p}_1)$. The pair $\{a_2, b_2\}$ is of Type $T_2 \in \{1, 2, \dots, k\}$. By our assumption, one can derive a \mathcal{P} -identity $\mathbf{p}_1 \approx \mathbf{p}_2$ from Δ so that for each $i > T_2$ we have $\text{Chaos}_i(\mathbf{p}_1 \approx \mathbf{v}) = \text{Chaos}_i(\mathbf{p}_2 \approx \mathbf{v})$ and $\text{Chaos}_{T_1}(\mathbf{p}_2 \approx \mathbf{v})$ is a proper subset of $\text{Chaos}_{T_1}(\mathbf{p}_1 \approx \mathbf{v})$. And so on.

If the sequence T_1, T_2, \dots contains number k then we are done. Otherwise, the sequence T_1, T_2, \dots must be infinite, because for each $j > 0$ we have $\text{Chaos}_k(\mathbf{p}_j \approx \mathbf{v}) = \text{Chaos}_k(\mathbf{u} \approx \mathbf{v})$. Let $m < k$ be the biggest number that repeats in this sequence infinite number of times. This means that starting with some number Q big enough, we do not see any critical pairs of Types bigger than m and that one can find a subsequence $Q < j_1 < j_2 < \dots$ so that $m = T_{j_1} = T_{j_2} = T_{j_3} = \dots$. Then for each $g = 1, 2, \dots$, the set $\text{Chaos}_m(\mathbf{p}_{j_g} \approx \mathbf{v})$ is a proper subset of $\text{Chaos}_m(\mathbf{p}_{j_{g-1}} \approx \mathbf{v})$. This means that the number of critical pairs of Type m must be decreasing to zero. A contradiction. \square

The desired statement immediately follows from the following.

Claim 2. *For each $0 < k \leq n$, every \mathcal{P} -identity of Type k can be derived from Δ and from a \mathcal{P} -identity of Type less than k .*

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be a \mathcal{P} -identity of Type k . Then by Claim 1, one can derive a \mathcal{P} -identity $\mathbf{u} \approx \mathbf{w}_1$ from Δ so that the identity $\mathbf{w}_1 \approx \mathbf{v}$ is also of Type k and contains a critical pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{w}_1)$ of Type k . Notice that $\text{Chaos}_k(\mathbf{w}_1 \approx \mathbf{v}) = \text{Chaos}_k(\mathbf{u} \approx \mathbf{v})$. By our assumption, one can derive a \mathcal{P} -identity $\mathbf{w}_1 \approx \mathbf{p}_1$ from Δ so that the identity $\mathbf{p}_1 \approx \mathbf{v}$ has Type at most k and $\text{Chaos}_k(\mathbf{p}_1 \approx \mathbf{v})$ is a proper subset of $\text{Chaos}_k(\mathbf{w}_1 \approx \mathbf{v})$. If $\text{Chaos}_k(\mathbf{p}_1 \approx \mathbf{v})$ is not empty, then by Claim 1, one can derive a \mathcal{P} -identity $\mathbf{p}_1 \approx \mathbf{w}_2$ from Δ so that the identity $\mathbf{w}_2 \approx \mathbf{v}$ is also of Type k and contains a critical pair $\{a, b\} \subseteq \text{OccSet}(\mathbf{p}_2)$ of Type k . Notice that $\text{Chaos}_k(\mathbf{w}_2 \approx \mathbf{v}) = \text{Chaos}_k(\mathbf{p}_1 \approx \mathbf{v})$. By our assumption, one can derive a \mathcal{P} -identity $\mathbf{w}_2 \approx \mathbf{p}_2$ from Δ so that the identity $\mathbf{p}_2 \approx \mathbf{v}$ is of Type at most k and $\text{Chaos}_k(\mathbf{p}_2 \approx \mathbf{v})$ is a proper subset of $\text{Chaos}_k(\mathbf{w}_2 \approx \mathbf{v})$. And so on. Eventually, for some $g < |\text{Chaos}_k(\mathbf{u} \approx \mathbf{v})|$, we obtain a \mathcal{P} -identity $\mathbf{p}_g \approx \mathbf{v}$ of Type less than k .

The sequence $\mathbf{u} \approx \mathbf{w}_1 \approx \mathbf{p}_1 \approx \mathbf{w}_2 \approx \mathbf{p}_2 \approx \dots \approx \mathbf{p}_g \approx \mathbf{v}$ gives us a derivation of $\mathbf{u} \approx \mathbf{v}$ from Δ and from a \mathcal{P} -identity of Type less than k . \square

\square

If x is a non-linear variable in a word $\mathbf{u} \in \mathfrak{A}^+$ then we define $Y(\mathbf{u}, x) \subseteq \text{OccSet}(\mathbf{u})$ as follows: for each $c \in \text{OccSet}(\mathbf{u})$ we have $c \in Y(\mathbf{u}, x)$ if and only if c is an occurrence of a variable other than x and there is a block \mathbf{B} in \mathbf{u} so that $1_{\mathbf{B}}x <_{\mathbf{u}} c <_{\mathbf{u}} \ell_{\mathbf{B}}x$. More generally, if x and y are two non-linear variables in a word \mathbf{u} then we define $Y(\mathbf{u}, x, y) \subseteq \text{OccSet}(\mathbf{u})$ as follows: for each $c \in \text{OccSet}(\mathbf{u})$ we have $c \in Y(\mathbf{u}, x, y)$ if and only if c is an occurrence of a variable other than x and y and there is a block \mathbf{B} in \mathbf{u} so that $a <_{\mathbf{u}} c <_{\mathbf{u}} b$, where a is the first occurrence of a variable in $\{x, y\}$ in \mathbf{B} and b is the last occurrence of a variable in $\{x, y\}$ in \mathbf{B} .

Notice that both sets $Y(\mathbf{u}, x)$ and $Y(\mathbf{u}, x, y)$ consist only of occurrences of non-linear variables in \mathbf{u} . For example, if $\mathbf{u} = yxxyt_1xyzxyxt_2yzxzx$ then $Y(\mathbf{u}, x) = \{3_{\mathbf{u}}y, 1_{\mathbf{u}}z, 4_{\mathbf{u}}y, 4_{\mathbf{u}}z\}$ and $Y(\mathbf{u}, x, y) = \{1_{\mathbf{u}}z, 3_{\mathbf{u}}z, 4_{\mathbf{u}}z\}$.

Evidently, set $Y(\mathbf{u}, x)$ is empty if and only if all occurrences of x in \mathbf{u} are collected together in each block of \mathbf{u} . Likewise, set $Y(\mathbf{u}, x, y)$ is empty if and only if all occurrences of x and y in \mathbf{u} are collected together in each block of \mathbf{u} . Now we illustrate how to use Lemma 3.2.

Lemma 3.3. *If a monoid S satisfies the identities $\{\sigma_\mu, \sigma_2\}$ (or dually, $\{\sigma_\mu, \sigma_1\}$), then all block-balanced identities of S can be derived from its block-balanced identities with two non-linear variables.*

Proof. Let \mathcal{P} be the property of being a block-balanced identity of S and Δ be the set of all block-balanced identities of S with two non-linear variables. We assign a Type to each pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of occurrences of distinct variables $x \neq y$ in a word \mathbf{u} as follows. If one of the variables $\{x, y\}$ is linear in \mathbf{u} then we say that $\{c, d\}$ is of Type 3. If both x and y are non-linear in \mathbf{u} , then we say that $\{c, d\}$ is of Type 2 if $\{c, d\} = \{1_{\mathbf{u}}x, 1_{\mathbf{u}}y\}$ and of Type 1 otherwise.

Let $\mathbf{u} \approx \mathbf{v}$ be a block-balanced identity of S and $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ be a critical pair in $\mathbf{u} \approx \mathbf{v}$. If $\{c, d\}$ is of Type 1, then by using an identity from $\{\sigma_\mu, \sigma_2\}^\delta$ we

swap c and d in \mathbf{u} and obtain a word \mathbf{w} . Evidently, the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

If $\{c, d\}$ is of Type 2, then $c = {}_{1\mathbf{u}}x$ and $d = {}_{1\mathbf{u}}y$. In this case we obtain the word \mathbf{w} required by Lemma 3.2 as follows.

First, we make the set $Y(\mathbf{u}, x, y)$ empty as follows. If the set $Y(\mathbf{u}, x, y)$ contains an occurrence p of some variable, then there is a block \mathbf{B} in \mathbf{u} so that $a <_{\mathbf{u}} p <_{\mathbf{u}} b$, where a is the first occurrence of $\{x, y\}$ in \mathbf{B} and b is the last occurrence of $\{x, y\}$ in \mathbf{B} . Since the pair $\{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ is adjacent in \mathbf{u} , b is a non-first occurrence of x or y . By using the identities $\{\sigma_\mu, \sigma_2\}^\delta$ and commuting adjacent occurrences of variables, we can move p to the right until we obtain a word \mathbf{w}_1 so that $b <_{\mathbf{w}_1} p$. Evidently, $|Y(\mathbf{w}_1, x, y)| < |Y(\mathbf{u}, x, y)|$. If the set $Y(\mathbf{w}_1, x, y)$ is still not empty, by using the identities $\{\sigma_\mu, \sigma_2\}^\delta$, we derive an identity $\mathbf{w}_1 \approx \mathbf{w}_2$ in a similar manner so that $|Y(\mathbf{w}_2, x, y)| < |Y(\mathbf{w}_1, x, y)|$. And so on. By repeating this procedure $n < |Y(\mathbf{u}, x, y)|$ times, we obtain a word \mathbf{w}_n so that the set $Y(\mathbf{w}_n, x, y)$ is empty. This means that all occurrences of x and y are collected together in each block of \mathbf{w}_n . Now we apply the identity $\mathbf{w}_n(x, y, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, y, \text{Lin}(\mathbf{u}))$ to \mathbf{w}_n and obtain the word \mathbf{w} so that both conditions of Lemma 3.2 are satisfied.

Evidently, a block-balanced identity does not contain any unstable pairs of Type 3. Since all the requirements of Lemma 3.2 are satisfied, all block-balanced identities of S can be derived from its block-balanced identities with two non-linear variables. \square

Lemma 3.4. *Suppose that every \mathcal{P}_1 -identity $\mathbf{u} \approx \mathbf{v}$ of a monoid S satisfies the following condition:*

() if for some variable $x \in \mathfrak{A}$ the identity $\mathbf{u}(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ is non-trivial and the set $Y(\mathbf{u}, x)$ is not empty, then by using some block-balanced identities of S one can derive an identity $\mathbf{u} \approx \mathbf{w}$ so that $|Y(\mathbf{w}, x)| < |Y(\mathbf{u}, x)|$.*

Then all \mathcal{P}_1 -identities of S can be derived from the almost-linear and block-balanced identities of S .

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be a \mathcal{P}_1 -identity of S and k denote the number of non-linear variables in \mathbf{u} . If the identity $\mathbf{u} \approx \mathbf{v}$ is not block-balanced, then for some variable x so that $\text{occ}_{\mathbf{u}}(x) > 1$ the identity $\mathbf{u}(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ is non-trivial. If the set $Y(\mathbf{u}, x)$ is not empty, then in view of Condition (*), by using some block-balanced identities of S one can derive an identity $\mathbf{u} \approx \mathbf{w}_1$ so that $|Y(\mathbf{w}_1, x)| < |Y(\mathbf{u}, x)|$. If the set $Y(\mathbf{w}_1, x)$ is still not empty, then we repeat the same arguments and obtain an identity $\mathbf{u} \approx \mathbf{w}_2$ so that $|Y(\mathbf{w}_2, x)| < |Y(\mathbf{w}_1, x)|$. After repeating this procedure at most $n < |Y(\mathbf{u}, x)|$ times we obtain an identity $\mathbf{u} \approx \mathbf{w}_n$ so that the set $Y(\mathbf{w}_n, x)$ is empty. This means that all occurrences of x are collected together in each block of \mathbf{w}_n .

Now the word $\mathbf{w}_n(x, \text{Lin}(\mathbf{u})) = \mathbf{u}(x, \text{Lin}(\mathbf{u}))$ is applicable to \mathbf{w}_n . So, for some word \mathbf{v}_1 we have $\mathbf{w}_n(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u})) \vdash \mathbf{w}_n \approx \mathbf{v}_1$. Notice that $\mathbf{v}_1(x, \text{Lin}(\mathbf{u})) = \mathbf{v}(x, \text{Lin}(\mathbf{u}))$. If the identity $\mathbf{v}_1 \approx \mathbf{v}$ is not block-balanced, then for some variable y the identity $\mathbf{v}_1(y, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(y, \text{Lin}(\mathbf{u}))$ is non-trivial. By repeating the same arguments, we obtain an identity $\mathbf{v}_2 \approx \mathbf{v}$ so that $\mathbf{v}_2(x, \text{Lin}(\mathbf{u})) = \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ and

$\mathbf{v}_2(y, \text{Lin}(\mathbf{u})) = \mathbf{v}(y, \text{Lin}(\mathbf{u}))$. By iterating this process at most k times, we obtain a block-balanced identity $\mathbf{v}_m \approx \mathbf{v}$ for some $m \leq k$. The sequence $\mathbf{u} \approx \mathbf{w}_1 \approx \mathbf{w}_2 \dots \mathbf{w}_{n(x)} \approx \mathbf{v}_1 \approx \mathbf{w}'_1 \dots \mathbf{w}'_{n(y)} \approx \mathbf{v}_2 \dots \approx \mathbf{v}_m \approx \mathbf{v}$ gives us a derivation of $\mathbf{u} \approx \mathbf{v}$ from some almost-linear and block-balanced identities of S . \square

Now we reprove the mentioned result of Lee.

Theorem 3.5. [4, Theorem 1.1] *Every monoid that satisfies the identities $\{\sigma_1, \sigma_\mu\}$ (or dually, $\{\sigma_\mu, \sigma_2\}$) is finitely based by some almost-linear identities and by some block-balanced identities with two non-linear variables.*

Proof. Let S be a monoid so that $S \models \{\sigma_1, \sigma_\mu\}$.

Claim 3. *All identities of S can be derived from the almost-linear and block-balanced identities of S .*

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be a \mathcal{P}_1 -identity of S . Suppose that for some variable $x \in \mathfrak{A}$ the identity $\mathbf{u}(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ is non-trivial and the set $Y(\mathbf{u}, x)$ is not empty. If c is the smallest in order $<_{\mathbf{u}}$ element of $Y(\mathbf{u}, x)$, then by using $\{\sigma_1, \sigma_\mu\}^\delta$ and commuting adjacent occurrences of variables, one can move c to the left and obtain a word \mathbf{w} so that $\ell_{\mathbf{B}}x <_{\mathbf{w}} c$. Then $|Y(\mathbf{w}, x)| < |Y(\mathbf{u}, x)|$. Since Condition (*) of Lemma 3.4 is satisfied, all \mathcal{P}_1 -identities of S can be derived from the almost-linear and block-balanced identities of S .

If the word xy is an isoterm for S , then every identity $\mathbf{u} \approx \mathbf{v}$ of S satisfies property \mathcal{P}_1 . If the word xy is not an isoterm for S , then in view of Fact 2.1, we may assume that $S \models x \approx x^n$ for some $n > 1$ and satisfies only regular identities. Then by using the identity $x \approx x^n$, one can transform every word \mathbf{p} into a word \mathbf{u} so that each variable is non-linear in \mathbf{u} . This means that every identity of S can be derived from $x \approx x^n$ and from a \mathcal{P}_1 -identity of S . In any case, all identities of S can be derived from the almost-linear and block-balanced identities of S . \square

By the result of Volkov (Lemma 2.2) all almost-linear identities of S can be derived from its finite subset. By Lemma 3.3, all block-balanced identities of S can be derived from its block-balanced identities with two non-linear variables. If \mathbf{u} is a word with two non-linear variables then by using the identities $\{\sigma_\mu, \sigma_2\}^\delta$ and commuting adjacent occurrences of variables, the word \mathbf{u} can be transform into one side of an identity of the form (1). By the result of Lee (Lemma 2.3), all identities of S of the form (1) can be derived from its finite subset. Therefore, the monoid S is finitely based by some almost-linear identities and by some block-balanced identities with two non-linear variables. \square

The following statement can be easily deduced either from Proposition 4.1 of [4] or from Claim 3.

Corollary 3.6. *If a monoid S satisfies the identities $\{\sigma_1, \sigma_\mu, \sigma_2\}$, then S is finitely based by $\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta$ and some almost-linear identities.*

Let W_{AL} denote the set of all almost-linear words. Corollary 3.6 immediately implies the following.

Corollary 3.7. $\text{var}\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta = \text{var}S(W_{AL})$.

4 Some finitely based monoids in $\text{var}\{\sigma_1, \sigma_2\}$ and in $\text{var}\{\sigma_\mu\}$

We use letter t with or without subscripts to denote linear (1-occurring) variables. If we use letter t several times in a word, we assume that different occurrences of t represent distinct linear variables. A word \mathbf{u} is said to be an *isoterm* ([6]) for a semigroup S if S does not satisfy any nontrivial identity of the form $\mathbf{u} \approx \mathbf{v}$.

Lemma 4.1. *Let S be a monoid so that $S \models \{\sigma_1, \sigma_2\}$. If for each $k > 1$, S satisfies the following property, then every identity of S can be derived from some almost-linear identities of S and from some block-balanced identities of S :*

(*) *If one of the words $\{x^k t, t x^k\}$ is not an isoterm for S then for some $0 < d < k$, S satisfies either $x^{k-d} t x^d y t y \approx x^{k-d} t x^{d-1} y x t y$ or $x t x y^d t y^{k-d} \approx x t y x y^{d-1} t y^{k-d}$.*

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be a \mathcal{P}_1 -identity of S . Let us check Condition (*) in Lemma 3.4. Suppose that for some variable $x \in \mathfrak{A}$ the identity $\mathbf{u}(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ is non-trivial and the set $Y(\mathbf{u}, x)$ is not empty. Since the set $Y(\mathbf{u}, x)$ is not empty, one can find a block \mathbf{B} in \mathbf{u} so that for some occurrence c of a variable $y \neq x$ we have ${}_{1\mathbf{B}}x <_{\mathbf{u}} c <_{\mathbf{u}} {}_{\ell\mathbf{B}}x$. Let c denote the smallest in order $<_{\mathbf{u}}$ element of $Y(\mathbf{u}, x)$ with this property and d denote the largest in order $<_{\mathbf{u}}$ element of $Y(\mathbf{u}, x)$ with this property. (So, d is an occurrence of some variable $z \neq x$.) If ${}_{1\mathbf{B}}x$ is not the first occurrence of x in \mathbf{u} then by using $\{\sigma_1, \sigma_2\}^\delta$ we move c to the left until we obtain a word \mathbf{w} so that $c \ll_{\mathbf{w}} {}_{1\mathbf{B}}x$. If ${}_{\ell\mathbf{B}}x$ is not the last occurrence of x in \mathbf{u} then by using $\{\sigma_1, \sigma_2\}^\delta$ we move d to the right until we obtain a word \mathbf{w} so that ${}_{\ell\mathbf{B}}x \ll_{\mathbf{w}} d$.

If ${}_{1\mathbf{B}}x = {}_{1\mathbf{u}}x$ and ${}_{\ell\mathbf{B}}x = {}_{\ell\mathbf{u}}x$, then all occurrences of x are in the same block \mathbf{B} of \mathbf{u} . Denote $\text{occ}_{\mathbf{u}}(x) = k$. Since the identity $\mathbf{u}(x, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, \text{Lin}(\mathbf{u}))$ is non-trivial, one of the words $x^k t$ or $t x^k$ is not an isoterm for S . Therefore, by Condition (*), S satisfies either $x^{k-d} t x^d y t y \approx x^{k-d} t x^{d-1} y x t y$ or $x t x y^d t y^{k-d} \approx x t y x y^{d-1} t y^{k-d}$. In view of the symmetry and the fact that S is a monoid, we may assume that S satisfies $x^k y t y \approx x^{k-1} y x t y$.

In this case, by using some identities in $\{\sigma_1, \sigma_2\}^\delta$ and moving occurrences of x other than ${}_{1\mathbf{u}}x$ and ${}_{\ell\mathbf{u}}x$ to the left toward the first occurrence of x , we obtain a word \mathbf{r} so that all occurrences of x except for ${}_{\ell\mathbf{r}}x$ are collected together in \mathbf{r} . Notice that $Y(\mathbf{u}, x) = Y(\mathbf{r}, x)$ and $d \ll_{\mathbf{r}} ({}_{\ell\mathbf{r}}x)$. If d is not the first occurrence of z then by using an identity in $\{\sigma_2\}^\delta$ we obtain a word \mathbf{w} so that $({}_{\ell\mathbf{r}}x) \ll_{\mathbf{r}} d$. If d is the first occurrence of z then by using the identity $x^{k-1} z t z \approx x^k z t z$ we obtain a word \mathbf{w} so that $({}_{\ell\mathbf{r}}x) \ll_{\mathbf{r}} d$.

In any case, we have $|Y(\mathbf{w}, x)| < |Y(\mathbf{u}, x)|$. Therefore, by Lemma 3.4, every \mathcal{P}_1 -identity of S can be derived from some almost-linear and block-balanced identities

of S . If the word xy is an isoterm for S , the monoid S satisfies only \mathcal{P}_1 -identities. If the word xy is not an isoterm for S , then by Condition (*), S satisfies σ_μ . Then by Corollary 3.6, S is finitely based by some almost-linear identities and by $\{\sigma_1, \sigma_2, \sigma_\mu\}$. In any case, every identity of S can be derived from some almost-linear and block-balanced identities of S . \square

For $n > 0$, a word \mathbf{u} is called n -limited if each variable occurs in \mathbf{u} at most n times. An identity is called n -limited if both sides of this identity are n -limited words.

Theorem 4.2. *Let S be a monoid so that $S \models \{\sigma_1, \sigma_2\}$. Suppose also that S satisfies one of the following conditions:*

- (i) *for each $k > 0$ the word $x^k y^k$ is an isoterm for S ;*
- (ii) *$S \models \{t_1 x t_2 x \dots t_{k+1} x \approx x^{k+1} t_1 t_2 \dots t_{k+1}, x^{k+1} \approx x^{k+2}\}$ where k is the maximal so that the word $x^k y^k$ is an isoterm for S .*

Then S is finitely based by some almost-linear identities together with $\{\sigma_1, \sigma_2\}^\delta$.

Proof. If S satisfies Condition (ii) then it is easy to see that every identity of S can be derived from $\{t_1 x t_2 x \dots t_{k+1} x \approx x^{k+1} t_1 t_2 \dots t_{k+1}, x^{k+1} \approx x^{k+2}\}^\delta$ and a k -limited identity of S . In view of Lemma 4.1, every k -limited identity of S can be derived from some (k -limited) almost-linear and (k -limited) block-balanced identities of S .

Claim 4. *Every k -limited block-balanced identity of S is a consequence of $\{\sigma_1, \sigma_2\}^\delta$.*

Proof. We assign a Type to each pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of occurrences of distinct variables in a word \mathbf{u} as follows. If $\{c, d\} = \{1_{\mathbf{u}}x, \ell_{\mathbf{u}}y\}$ for some variables $x \neq y$ then we say that $\{c, d\}$ is of Type 2. Otherwise, $\{c, d\}$ is of Type 1.

Let $\mathbf{u} \approx \mathbf{v}$ be a k -limited block-balanced identity of S and $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ be a critical pair in $\mathbf{u} \approx \mathbf{v}$. Since each letter occurs in \mathbf{u} at most k times and $x^k y^k$ is an isoterm for S , the identity $\mathbf{u} \approx \mathbf{v}$ does not contain any unstable pairs of Type 2. Suppose that $\{c, d\}$ is of Type 1. Then by using an identity from $\{\sigma_1, \sigma_2\}^\delta$ we swap c and d in \mathbf{u} and obtain a word \mathbf{w} . Evidently, the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2. Therefore, by Lemma 3.2, every k -limited block-balanced identity of S can be derived from $\{\sigma_1, \sigma_2\}^\delta$. \square

If S satisfies Condition (i) then the proof is similar but more simple. \square

Theorem 4.2(i) immediately implies the following statement similar to Corollary 3.7.

Corollary 4.3. $\text{var}\{\sigma_1, \sigma_2\}^\delta = \text{var}S(W_{AL} \cup \{a^k b^k \mid k > 0\})$.

We say that a pair of variables $\{x, y\}$ is b -unstable in a word \mathbf{u} with respect to a semigroup S if S satisfies a block-balanced identity of the form $\mathbf{u} \approx \mathbf{v}$ so that $\mathbf{u}(x, y) \neq \mathbf{v}(x, y)$. The following theorem generalizes Theorem 4.2(ii) into a more sophisticated condition under which a monoid is finitely based.

Theorem 4.4. *Let S be a monoid so that $S \models \{\sigma_1, \sigma_2\}$. Let $m > 0$ be the maximal so that the word $x^m y^m$ is an isoterm for S . Suppose that for some $0 < d \leq m$, S satisfies either $x^{m+1-d} t x^d y t y \approx x^{m+1-d} t x^{d-1} y x t y$ or $x t x y^d t y^{m+1-d} \approx x t y x y^{d-1} t y^{m+1-d}$. Suppose also that for each $1 < k \leq m$, S satisfies each of the following dual conditions:*

(i) *If for some almost-linear word $\mathbf{A}x$ with $\text{occ}_{\mathbf{A}}(x) > 0$ the pair $\{x, y\}$ is b-unstable in $\mathbf{A}x y^k$ with respect to S then for some $0 < c < k$, S satisfies the identity $\mathbf{A}x y^c t y^{k-c} \approx \mathbf{A}y x y^{c-1} t y^{k-c}$;*

(ii) *If for some almost-linear word $y\mathbf{B}$ with $\text{occ}_{\mathbf{B}}(y) > 0$ the pair $\{x, y\}$ is b-unstable in $x^k y\mathbf{B}$ with respect to S then for some $0 < p < k$, S satisfies the identity $x^{k-p} t x^p y \mathbf{B} \approx x^{k-p} t x^{p-1} y x \mathbf{B}$.*

Then S is finitely based by some almost-linear identities and by some block-balanced identities with two non-linear variables.

Proof. If $m = 1$ then $S \models \sigma_\mu$ and by Corollary 3.6, the monoid S is finitely based by some almost-linear identities and by $\{\sigma_1, \sigma_2, \sigma_\mu\}$.

So, we may assume that $m > 1$. Then by Condition (i), for some $0 < d \leq m$, S satisfies either $x^{m+1-d} t x^d y t y \approx x^{m+1-d} t x^{d-1} y x t y$ or $x t x y^d t y^{m+1-d} \approx x t y x y^{d-1} t y^{m+1-d}$. Since all conditions are symmetric, without loss of generality we may assume that S satisfies $x^{m+1-d} t x^d y t y \approx x^{m+1-d} t x^{d-1} y x t y$.

Claim 5. *Every block-balanced identity of S can be derived from some block-balanced identities of S with two non-linear variables.*

Proof. We assign a Type to each pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of occurrences of distinct variables in a word \mathbf{u} as follows. If $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) \leq m$ then we say that $\{c, d\}$ is of Type 3. If $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) > m$ and there is a linear letter in \mathbf{u} between ${}_{1\mathbf{u}}x$ and $\ell_{\mathbf{u}}x$ then we say that $\{c, d\}$ is also of Type 3. If $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) > m$ and there is no linear letter in \mathbf{u} between ${}_{1\mathbf{u}}x$ and $\ell_{\mathbf{u}}x$ then we say that $\{c, d\}$ is of Type 2. Otherwise, $\{c, d\}$ is of Type 1.

Let $\mathbf{u} \approx \mathbf{v}$ be a block-balanced identity of S and $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ be a critical pair in $\mathbf{u} \approx \mathbf{v}$. Suppose that $\{c, d\}$ is of Type 1. Then by using an identity from $\{\sigma_1, \sigma_2\}^\delta$ we swap c and d in \mathbf{u} and obtain a word \mathbf{w} . Evidently, the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Now suppose that $\{c, d\}$ is of Type 2. Then $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) = n > m$ and there is no linear letter in \mathbf{u} between ${}_{1\mathbf{u}}x$ and $\ell_{\mathbf{u}}x$. In this case, by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{r} so that all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{1_{\mathbf{r}}x, 2_{\mathbf{r}}x, \dots, (n-d)_{\mathbf{r}}x\}$ and all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{(n-d+1)_{\mathbf{r}}x, \dots, n_{\mathbf{r}}x, 1_{\mathbf{r}}y\}$ are collected together. After that by using an identity in $\{x^{m+1-d} t x^d y t y \approx x^{m+1-d} t x^{d-1} y x t y\}^\delta$ we swap c and d in \mathbf{r} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Finally, suppose that $\{c, d\}$ is of Type 3 and consider two cases.

Case 1: $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_1\mathbf{u}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) > m$ and there is a linear letter in \mathbf{u} between ${}_1\mathbf{u}x$ and $\ell_{\mathbf{u}}x$. If there is a linear letter between ${}_1\mathbf{u}y$ and $\ell_{\mathbf{u}}y$, then by using $\{\sigma_1, \sigma_2\}^\delta$ we collect all occurrences of x and y in each block together and obtain a word \mathbf{s} . After that we apply the identity $\mathbf{s}(x, y, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, y, \text{Lin}(\mathbf{u}))$ to \mathbf{s} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Now suppose that there is no linear letter between ${}_1\mathbf{u}y$ and $\ell_{\mathbf{u}}y$. Let \mathbf{A} be an almost-linear word so that $\mathbf{u}(x, \text{Lin}(\mathbf{u})) = \mathbf{A}x$. Denote $\text{occ}_{\mathbf{u}}(y) = k$ and consider two cases. If $k \leq m$, then by Condition (i), S satisfies the identity $\mathbf{A}xy^c ty^{k-c} \approx \mathbf{A}yxy^{c-1}ty^{k-c}$ for some $0 < c < k$. In this case, by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{r} so that all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{\ell_{\mathbf{r}}x, {}_1\mathbf{r}y, {}_2\mathbf{r}y, \dots, {}_c\mathbf{r}y\}$ and all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{({}_{c+1})\mathbf{r}y, ({}_{c+2})\mathbf{r}y, \dots, {}_k\mathbf{r}y\}$ are collected together. After that, we apply the identity $\mathbf{A}xy^c ty^{k-c} \approx \mathbf{A}yxy^{c-1}ty^{k-c}$ to \mathbf{r} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Now assume that $k > m$. In this case, we collect all occurrences of y together as follows. First, by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{r} so that all the elements of $\text{OccSet}(\mathbf{u})$ which are in the set $\{\ell_{\mathbf{r}}x, {}_1\mathbf{r}y, {}_2\mathbf{r}y, \dots, ({}_{k-1})\mathbf{r}y\}$ are collected together. If $({}_{k-1})\mathbf{r}y$ and ${}_k\mathbf{r}y$ are not adjacent in \mathbf{r} then one can find an occurrence p of some non-linear variable $z \notin \{x, y\}$ so that $p \ll_{\mathbf{r}} ({}_k\mathbf{r}y)$. If p is not the first occurrence of z then by using an identity in $\{\sigma_2\}^\delta$, we obtain a word \mathbf{s} so that $({}_k\mathbf{s}y) \ll_{\mathbf{s}} p$. Notice that $|Y(\mathbf{s}, x)| < |Y(\mathbf{u}, x)|$. If p is the first occurrence of z then first, by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{q} so that all the elements of $\text{OccSet}(\mathbf{q})$ which are in the set $\{({}_{k-d+1})\mathbf{q}y, \dots, ({}_{k-1})\mathbf{q}y, p, {}_k\mathbf{q}y\}$ are collected together. After that, by using an identity in $\{y^{m+1-d}ty^d ztz \approx y^{m+1-d}ty^{d-1}zytz\}^\delta$, we obtain a word \mathbf{s} so that $({}_k\mathbf{s}y) \ll_{\mathbf{s}} p$. Notice that $|Y(\mathbf{s}, x)| < |Y(\mathbf{u}, x)|$. Eventually, we obtain a word \mathbf{t} so that all the elements of $\text{OccSet}(\mathbf{t})$ which are in the set $\{\ell_{\mathbf{t}}x, {}_1\mathbf{t}y, {}_2\mathbf{t}y, \dots, {}_k\mathbf{t}y\}$ are collected together.

After that we apply the identity $\mathbf{t}(x, y, \text{Lin}(\mathbf{u})) = \mathbf{A}xy^k \approx \mathbf{v}(x, y, \text{Lin}(\mathbf{u}))$ to \mathbf{t} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Case 2: $\{c, d\} = \{\ell_{\mathbf{u}}x, {}_1\mathbf{u}y\}$ for some variables $x \neq y$ so that $\text{occ}_{\mathbf{u}}(x) = n \leq m$. Denote $\text{occ}_{\mathbf{u}}(y) = k$. Since the word $x^m y^m$ is an isotherm for S , we have $k > m$.

If ${}_1y$ and ℓy are in the same block in \mathbf{u} , then as in the previous case, we obtain a word \mathbf{t} so that all the elements of $\text{OccSet}(\mathbf{t})$ which are in the set $\{\ell_{\mathbf{t}}x, {}_1\mathbf{t}y, {}_2\mathbf{t}y, \dots, {}_k\mathbf{t}y\}$ are collected together.

If ${}_1x$ and ℓx are not in the same block in \mathbf{t} , then by using $\{\sigma_1, \sigma_2\}^\delta$ we collect all occurrences of x in each block together and obtain a word \mathbf{q} . After that we apply the identity $\mathbf{q}(x, y, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, y, \text{Lin}(\mathbf{u}))$ to \mathbf{q} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Since $n \leq m$, by Condition (ii), S satisfies the identity $x^{n-p}tx^py^k \approx x^{n-p}tx^{p-1}yxy^{k-1}$ for some $0 < p < n$. If ${}_1x$ and ℓx are in the same block in \mathbf{t} , then by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{r} so that all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{{}_1\mathbf{r}x, {}_2\mathbf{r}x, \dots, ({}_{n-p})\mathbf{r}x\}$ and all the elements of $\text{OccSet}(\mathbf{r})$ which are in the

set $\{(n-p+1)\mathbf{r}x, \dots, n\mathbf{r}x, 1\mathbf{r}y\}$ are collected together. After that by using $x^{n-p}tx^py^k \approx x^{n-p}tx^{p-1}yxy^{k-1}$ we swap c and d in \mathbf{r} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Now assume that $1y$ and ℓy are not in the same block in \mathbf{u} . If $1x$ and ℓx also are not in the same block in u , then by using $\{\sigma_1, \sigma_2\}^\delta$ we collect all occurrences of x and y in each block together and obtain a word \mathbf{s} . After that we apply the identity $\mathbf{s}(x, y, \text{Lin}(\mathbf{u})) \approx \mathbf{v}(x, y, \text{Lin}(\mathbf{u}))$ to \mathbf{s} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

It is left to assume that $1y$ and ℓy are not in the same block in \mathbf{u} , but $1x$ and ℓx are in the same block in \mathbf{u} . Let \mathbf{B} be an almost-linear word so that $\mathbf{u}(y, \text{Lin}(\mathbf{u})) = y\mathbf{B}$. Since the pair $\{x, y\}$ is unstable in $x^n y\mathbf{B}$ with respect to S , by Condition (ii), S satisfies the identity $x^{n-p}tx^py\mathbf{B} \approx x^{n-p}tx^{p-1}yx\mathbf{B}$. In this case, by using $\{\sigma_1, \sigma_2\}^\delta$ we obtain a word \mathbf{r} so that all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{1\mathbf{r}x, 2\mathbf{r}x, \dots, (n-p)\mathbf{r}x\}$ and all the elements of $\text{OccSet}(\mathbf{r})$ which are in the set $\{(n-p+1)\mathbf{r}x, \dots, n\mathbf{r}x, 1\mathbf{r}y\}$ are collected together. After that by using $x^{n-p}tx^py\mathbf{B} \approx x^{n-p}tx^{p-1}yx\mathbf{B}$ we swap c and d in \mathbf{r} and obtain a word \mathbf{w} . It is easy to see that the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2. \square

By Lemma 4.1, every identity of S can be derived from some almost-linear identities of S and from some block-balanced identities of S . By the result of Volkov (Fact 2.2) all almost-linear identities of S can be derived from its finite subset. By Claim 5, every block-balanced identity of S can be derived from some block-balanced identities of S with two non-linear variables. If \mathbf{u} is a word with two non-linear variables then by using the identities $\{\sigma_1, \sigma_2\}^\delta$ and commuting adjacent occurrences of variables, the word \mathbf{u} can be transform into one side of an identity of the following form:

$$\begin{aligned} x^{\alpha_0}t_1x^{\alpha_1}t_2 \dots x^{\alpha_{n-1}}t_nx^{\alpha_n}y^{\beta_m}p_my^{\alpha_{m-1}} \dots y^{\beta_3}p_2y^{\beta_1}p_1y^{\beta_0} \approx \\ x^{\alpha_0}t_1x^{\alpha_1}t_2 \dots x^{\alpha_{n-1}}t_ny^{\beta_m}x^{\alpha_n}p_my^{\alpha_{m-1}} \dots y^{\beta_3}p_2y^{\beta_1}p_1y^{\beta_0}, \end{aligned}$$

where $n, m, \alpha_n + \beta_m > 0$ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_m \geq 0$.

By using the same arguments as in the proof of Proposition 5.7 in [4] (see Lemma 2.3 above) one can show that in the presence of $\{\sigma_1, \sigma_2\}^\delta$, every set of identities of this form can be derived from its finite subset. Therefore, the monoid S is finitely based by some almost-linear identities and by some block-balanced identities with two non-linear variables. \square

Example 4.5. Let W be a set of words of the form $a_1^{\alpha_1} \dots a_m^{\alpha_m}$ for some letters a_1, \dots, a_m and numbers $\alpha_1, \dots, \alpha_m$. Then monoid $S(W)$ is finitely based.

Proof. It is easy to check that $S(W)$ satisfies all conditions of Theorem 4.4. \square

We say that a 2-limited word \mathbf{u} is a xx -word if for each variable x with $\text{occ}_{\mathbf{u}}(x) = 2$, either $1_{\mathbf{u}}x \ll_{\mathbf{u}} 2_{\mathbf{u}}x$ or there is a linear letter in \mathbf{u} between $1_{\mathbf{u}}x$ and $2_{\mathbf{u}}x$. The next lemma is needed only to prove Theorem 4.8.

Lemma 4.6. *Every 2-limited word is equivalent to a xx -word modulo $\{\sigma_\mu, yxxy \approx xxyty\}^\delta$.*

Proof. Let \mathbf{u} be a 2-limited word. We say that a 2-occurring variable is a \mathcal{L} -variable in \mathbf{u} if there is no linear letters between ${}_{1\mathbf{u}}x$ and ${}_{2\mathbf{u}}x$. We use $\mathfrak{Q}(\mathbf{u}, x)$ to denote the set of all \mathcal{L} -variables $y \neq x$ so that both occurrences of y are between ${}_{1\mathbf{u}}x$ and ${}_{2\mathbf{u}}x$. If x is a \mathcal{L} -variable and $\mathfrak{Q}(\mathbf{u}, x) = \{z_1, \dots, z_m\}$ for some $m \geq 0$, then $Y(\mathbf{u}, x) = Y_1 \cup Y_2 \cup \{{}_{1\mathbf{p}}z_1, {}_{2\mathbf{p}}z_1, \dots, {}_{1\mathbf{p}}z_m, {}_{2\mathbf{p}}z_m\}$ where each element of Y_1 is the first occurrence of some variable in \mathbf{u} and each element of Y_2 is the second occurrence of some variable in \mathbf{u} . The desired statement follows immediately from the following.

Claim 6. *Every 2-limited word \mathbf{u} is equivalent modulo $\{\sigma_\mu, yxxy \approx xxyty\}^\delta$ to a word \mathbf{p} with the property that for each $m \geq 0$ and for each \mathcal{L} -variable x with $|\mathfrak{Q}(\mathbf{u}, x)| \leq m$ each of the following is true:*

- (i) ${}_{1\mathbf{p}}x \ll_{\mathbf{p}} {}_{2\mathbf{p}}x$;
- (ii) for each $c \in \text{OccSet}(\mathbf{u})$ we have $c <_{\mathbf{u}} {}_{1\mathbf{u}}x$ iff $c <_{\mathbf{p}} {}_{1\mathbf{p}}x$;
- (iii) for each $c \in \text{OccSet}(\mathbf{u})$ we have ${}_{2\mathbf{u}}x <_{\mathbf{u}} c$ iff ${}_{2\mathbf{p}}x <_{\mathbf{p}} c$.

Proof. First, we prove the statement for $m = 0$. Let x be a \mathcal{L} -variable in \mathbf{u} so that the set $\mathfrak{Q}(\mathbf{u}, x)$ is empty. Then $Y(\mathbf{u}, x) = Y_1 \cup Y_2$. If q' is the smallest in order $<_{\mathbf{u}}$ element in Y_2 , then by using the identities in $\{\sigma_\mu\}^\delta$ and commuting the adjacent occurrences of variables, we move q' to the left until we obtain a word \mathbf{s}_1 so that $q' \ll_{\mathbf{s}_1} {}_{1\mathbf{s}_1}x$. And so on. After repeating this $k = |Y_2|$ times, we obtain a word \mathbf{s}_k so that each occurrence of each variable between ${}_{1\mathbf{s}_k}x$ and ${}_{2\mathbf{s}_k}x$ is the first occurrence of this variable. Now by using the identities in $\{\sigma_\mu\}^\delta$ and commuting the adjacent occurrences of variables, we move ${}_{2\mathbf{s}_k}x$ to the left until we obtain a word \mathbf{r}_1 so that ${}_{1\mathbf{r}_1}x \ll_{\mathbf{r}_1} {}_{2\mathbf{r}_1}x$. Since we only “push out” the elements of $\text{OccSet}(\mathbf{u})$ which are between ${}_{1\mathbf{u}}x$ and ${}_{2\mathbf{u}}x$, the word \mathbf{r}_1 satisfies Properties (ii)-(iii) as well.

If $z \neq x$ is another \mathcal{L} -variable in \mathbf{u} so that the set $\mathfrak{Q}(\mathbf{u}, x)$ is empty, then by repeating the same procedure, we obtain a word \mathbf{r}_2 so that ${}_{1\mathbf{r}_2}x \ll_{\mathbf{r}_2} {}_{2\mathbf{r}_2}x$, ${}_{1\mathbf{r}_2}z \ll_{\mathbf{r}_2} {}_{2\mathbf{r}_2}z$ and Properties (ii)-(iii) are satisfied for x and z . And so on. Thus, the base of induction is established.

Let x be a \mathcal{L} -variable in \mathbf{u} with $\mathfrak{Q}(\mathbf{u}, x) = \{z_1, \dots, z_m\}$. By our induction hypothesis, the word \mathbf{u} is equivalent modulo $\{\sigma_\mu, yxxy \approx xxyty\}^\delta$ to a word \mathbf{p} with the property that for each $i = 1, \dots, m$ we have ${}_{1\mathbf{p}}x <_{\mathbf{p}} {}_{1\mathbf{p}}z_i \ll_{\mathbf{p}} {}_{2\mathbf{p}}z_i <_{\mathbf{p}} {}_{2\mathbf{p}}x$. If q' is the smallest in order $<_{\mathbf{p}}$ element in $Y_2 \cup \{{}_{1\mathbf{p}}z_1, {}_{2\mathbf{p}}z_1, \dots, {}_{1\mathbf{p}}z_m, {}_{2\mathbf{p}}z_m\}$, then we do the following. If $q' \in Y_2$ then by using the identities in $\{\sigma_\mu\}^\delta$ and commuting the adjacent occurrences of variables, we move q' to the left until we obtain a word \mathbf{s}_1 so that $q' \ll_{\mathbf{s}_1} {}_{1\mathbf{s}_1}x$. If $q' = {}_{1\mathbf{p}}z_i$ for some $i = 1, \dots, m$, then by using the identities in $\{yxxy \approx xxyty\}^\delta$, we move $({}_{1\mathbf{p}}z_i)({}_{2\mathbf{p}}z_i)$ to the left until we obtain a word \mathbf{s}_1 so that $({}_{1\mathbf{s}_1}z_i) \ll_{\mathbf{s}_1} ({}_{2\mathbf{s}_1}z_i) \ll_{\mathbf{s}_1} {}_{1\mathbf{s}_1}x$. And so on. After repeating this $k = |Y_2| + m$ times, we obtain a word \mathbf{s}_{k+m} so that each occurrence of each variable between ${}_{1\mathbf{s}_{k+m}}x$ and ${}_{2\mathbf{s}_{k+m}}x$ is the first occurrence of this variable. Now by using the identity σ_μ and commuting the adjacent occurrences of variables, we move ${}_{2\mathbf{s}_{k+m}}x$ to the left until we obtain a word \mathbf{r}_1 so that ${}_{1\mathbf{r}_1}x \ll_{\mathbf{r}_1} {}_{2\mathbf{r}_1}x$.

If $z \neq x$ is another \mathcal{L} -variable in \mathbf{u} with $\Omega(\mathbf{u}, x) = m$, then we repeat the same procedure and obtain a word \mathbf{r}_2 so that ${}_{1\mathbf{r}_2}x \ll_{\mathbf{r}_2} {}_{2\mathbf{r}_2}x$, ${}_{1\mathbf{r}_2}z \ll_{\mathbf{r}_2} {}_{2\mathbf{r}_2}z$ and Properties (ii)-(iii) are satisfied for x and z . And so on. Thus, the step of induction is established. \square

\square

Fact 4.7. (i) If the word $xytyx$ is an isoterm for a monoid S then the words $xyztzxy$ and $yzxtzyx$ can form an identity of S only with each other.

(ii) The word $xyztzxy$ is an isoterm for a monoid S if and only if the word $yzxtzyx$ is an isoterm for S .

Proof. (i) If S satisfies an identity $xyztzxy \approx \mathbf{u}$ then we have $\mathbf{u}(y, z, t) = yztzy$. If $\mathbf{u} \neq xyztzxy$ then the only possibility for \mathbf{u} is $yzxtzyx$.

Part (ii) immediately follows from part (i). \square

We say that an identity $\mathbf{u} \approx \mathbf{v}$ is a xx -identity if both \mathbf{u} and \mathbf{v} are xx -words. Part (i) of the following statement generalizes Theorem 3.2 in [2] which says that monoid $S(\{abtab, abtba\})$ is finitely based.

Theorem 4.8. Let S be a monoid so that $S \models \{t_1xt_2xt_3x \approx x^3t_1t_2t_3, x^3 \approx x^4, \sigma_\mu, yxxtxy \approx xxyty\} = \Omega$. Suppose also that S satisfies one of the following conditions:

(i) both words $xytyx$ and $xytxy$ are isoterns for S ;

(ii) the word $xyztzxy$ is an isoterm for S .

Then S is finitely based by a subset of $\Omega \cup \{ytyxx \approx yttxy, xxt \approx txx, xytxy \approx yxtyx\}^\delta$.

Proof. Let Δ denote the subset of $\{\sigma_\mu, yxxtxy \approx xxyty, ytyxx \approx yttxy, xytxy \approx yxtyx, xxt \approx txx\}^\delta$ satisfied by S . We use Lemma 3.2 to show that every xx -identity of S is a consequence of Δ .

We assign a Type to each pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of occurrences of distinct variables $x \neq y$ in a word \mathbf{u} as follows. If one of the variables $\{x, y\}$ occurs more than twice in \mathbf{u} then we say that $\{c, d\}$ is of Type 3. If both x and y are 2-occurring, $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}x\}$ or $\{c, d\} = \{{}_{1\mathbf{u}}y, {}_{1\mathbf{u}}y\}$ and there is a linear letter (possibly the same) between ${}_{1\mathbf{u}}x$ and ${}_{2\mathbf{u}}x$ and between ${}_{1\mathbf{u}}y$ and ${}_{2\mathbf{u}}y$ then we say that $\{c, d\}$ is of Type 2. Otherwise, $\{c, d\}$ is of Type 1.

Let $\mathbf{u} \approx \mathbf{v}$ be a xx -identity of S and $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ be a critical pair in $\mathbf{u} \approx \mathbf{v}$. Suppose that $\{c, d\}$ is of Type 1.

First assume that, say c is the only occurrence of a linear variable t in \mathbf{u} . Then, since the word txx is an isoterm for S , d must be an occurrence of a 2-occurring variable x and $\mathbf{u}(x, t) \approx \mathbf{v}(x, t)$ is the following identity: $xtt \approx txx$. Since ${}_{1\mathbf{u}}x \ll_{\mathbf{u}} {}_{2\mathbf{u}}x$, we can apply $xtt \approx txx$ to \mathbf{u} and obtain the word \mathbf{w} . Evidently, the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Next assume that $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{2\mathbf{u}}y\}$ for some 2-occurring variables x and y . Then by using an identity from $\{\sigma_\mu\}^\delta$ we swap c and d in \mathbf{u} and obtain a word \mathbf{w} . Evidently, the identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Now assume that $c = {}_{1\mathbf{u}}x \ll_{\mathbf{u}} {}_{1\mathbf{u}}y = d$ for some 2-occurring variables x and y . Let a denote the smallest in order $<_{\mathbf{u}}$ element of the set $\{{}_{2\mathbf{u}}x, {}_{2\mathbf{u}}y\}$. Since $\{c, d\}$ is of Type 1, there is no linear letter between ${}_{1\mathbf{u}}y$ and a . Since both \mathbf{u} and \mathbf{v} are xx -words, we have that $a = {}_{2\mathbf{u}}y$, $({}_{1\mathbf{u}}x) \ll_{\mathbf{u}} ({}_{1\mathbf{u}}y) \ll_{\mathbf{u}} ({}_{2\mathbf{u}}y)$ and $({}_{1\mathbf{v}}y) \ll_{\mathbf{v}} ({}_{2\mathbf{v}}y)$. We use the identity $xyytx \approx yyxtx$ and obtain the word \mathbf{w} so that $({}_{1\mathbf{w}}y) \ll_{\mathbf{w}} ({}_{2\mathbf{w}}y) \ll_{\mathbf{w}} ({}_{1\mathbf{w}}x)$. The identity $\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2.

Finally, assume that $c = {}_{2\mathbf{u}}x \ll_{\mathbf{u}} {}_{2\mathbf{u}}y = d$ for some 2-occurring variables x and y . Let b denote the largest in order $<_{\mathbf{u}}$ element of the set $\{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}y\}$. Since $\{c, d\}$ is of Type 1, there is no linear letter between b and ${}_{2\mathbf{u}}x$. Since \mathbf{u} is a xx -word, we have that $b = {}_{1\mathbf{q}}x$, $({}_{1\mathbf{u}}x) \ll_{\mathbf{u}} ({}_{2\mathbf{u}}x) \ll_{\mathbf{u}} ({}_{2\mathbf{u}}y)$, $({}_{1\mathbf{v}}x) \ll_{\mathbf{v}} ({}_{2\mathbf{v}}x)$ and there is a linear letter between ${}_{1\mathbf{u}}y$ and ${}_{1\mathbf{u}}x$. We apply the identity $yttxy = \mathbf{u}(x, y, t) \approx \mathbf{v}(x, y, t) = ytyxx$ to \mathbf{u} and obtain the word \mathbf{w} so that both conditions of Lemma 3.2 are satisfied.

If S satisfies Condition (i) which says that both words $xytyx$ and $xytxy$ are isoterm for S , then the identity $\mathbf{u} \approx \mathbf{v}$ does not have any unstable pairs of Type 2 and we are done.

Let us suppose that S satisfies Condition (ii) which says that the word $xyzttxzy$ is an isotherm for S . If $\{c, d\}$ is of Type 2, then $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ or $\{c, d\} = \{{}_{2\mathbf{u}}x, {}_{2\mathbf{u}}y\}$ for some 2-occurring variables $x \neq y$ and there is a linear letter between ${}_{1\mathbf{u}}x$ and ${}_{2\mathbf{u}}x$ and between ${}_{1\mathbf{u}}y$ and ${}_{2\mathbf{u}}y$. Since the word $xytyx$ is an isotherm for S , for some letter t we have $\mathbf{u}(x, y, t) = xytxy$ and $\mathbf{v}(x, y, t) = ytxyx$.

In view of the symmetry, without loss of generality, we may assume that $c = {}_{1\mathbf{u}}x \ll_{\mathbf{u}} {}_{1\mathbf{u}}y = d$. Since the word xyt_1xt_2y is an isotherm for S , there is no linear letter in \mathbf{u} between ${}_{2\mathbf{u}}x$ and ${}_{2\mathbf{u}}y$.

Claim 7. *If for some variable z we have ${}_{2\mathbf{u}}x <_{\mathbf{u}} {}_{2\mathbf{u}}z <_{\mathbf{u}} {}_{2\mathbf{u}}y$ then we have ${}_{2\mathbf{u}}x <_{\mathbf{u}} {}_{1\mathbf{u}}z \ll_{\mathbf{u}} {}_{2\mathbf{u}}z <_{\mathbf{u}} {}_{2\mathbf{u}}y$.*

Proof. If there is a linear letter between ${}_{1\mathbf{u}}z$ and ${}_{2\mathbf{u}}z$ then for some letter t we have $\mathbf{u}(x, y, z, t) = xyztxzy$ or $\mathbf{u}(x, y, z, t) = zxytxzy$. But by Fact 4.7, both these words are isoterm for S . The rest follows from the fact that \mathbf{u} is a xx -word. \square

In view of Claim 7 we have $Y(\mathbf{u}, x, y) = Y_1 \cup \{{}_{1\mathbf{p}}z_1, {}_{2\mathbf{p}}z_1, \dots, {}_{1\mathbf{p}}z_m, {}_{2\mathbf{p}}z_m\}$. If $m > 0$ then it is easy to see that S satisfies the identity $ytyxx \approx ytxxy$. Suppose that the set $Y(\mathbf{u}, x, y)$ is not empty and q is the smallest in order $<_{\mathbf{u}}$ element in $Y(\mathbf{u}, x, y)$. If $q \in Y_1$, we use $\{\sigma_\mu\}^\delta$ and obtain a word \mathbf{r}_1 so that $q \ll_{\mathbf{r}_1} {}_{2\mathbf{r}_1}x$. If q is the first occurrence of z_i for some $i = 1, \dots, m$, then we use $ytyxx \approx ytxxy$ and obtain a word \mathbf{r}_1 so that ${}_{1\mathbf{p}}z_1 \ll_{\mathbf{r}_1} {}_{2\mathbf{p}}z_1 \ll_{\mathbf{r}_1} {}_{2\mathbf{u}}x$. In any case we have $|Y(\mathbf{u}, x, y)| < |Y(\mathbf{r}_1, x, y)|$. And so on. After $m = |Y(\mathbf{u}, x, y)|$ steps we obtain a word \mathbf{r}_m so that the set $Y(\mathbf{r}_m, x, y)$ is empty. This means that ${}_{2\mathbf{u}}x \ll_{\mathbf{r}_m} {}_{2\mathbf{u}}y$. Now we apply the identity $xytxy \approx ytxyx$ to \mathbf{r}_m and obtain a word \mathbf{w} . It is easy to check that both conditions of Lemma 3.2 are satisfied.

Since, xx -identities do not have any unstable pairs of Type 3, by Lemma 3.2, every xx -identity of S can be derived from Δ . In view of Lemma 4.6, every 2-limited identity of S can be derived from some xx -identities of S . Finally, every identity of S can be derived from $\{t_1xt_2xt_3x \approx x^3t_1t_2t_3, x^3 \approx x^4\}^\delta$ and a 2-limited identity of

S . Therefore, every identity of S can be derived from a subset of $\Delta \cup \{t_1xt_2xt_3x \approx x^3t_1t_2t_3, x^3 \approx x^4\}^\delta = \Omega \cup \{ytyxx \approx ytxxy, xxt \approx txx, xytxy \approx yxtyx\}^\delta$. \square

Example 4.9. *The monoids $S(abctacb)$ and $S(cbatbca)$ are equationally equivalent and finitely based.*

Proof. These monoids are equationally equivalent by Fact 4.7 and finitely based by Theorem 4.8(ii). \square

5 Some derivation-stable properties of identities and a description of the equational theories for some varieties

We say that a property of identities \mathcal{P} is *derivation-stable* if an identity τ satisfies property \mathcal{P} whenever $\Sigma \vdash \tau$ and each identity in Σ satisfies property \mathcal{P} . It is easy to check that such properties of identities as being a balanced identity, being a regular identity, being a \mathcal{P}_n -identity, being a block-balanced identity are all derivation stable.

Let ϵ denote the empty word, W_n denote the set of all n -limited words in a two letter alphabet and W_{AL} denote the set of all almost-linear words. Notice that $S(\{\epsilon\})$ is isomorphic to the two-element semilattice and if $a \in \mathfrak{A}$ then $S(\{a^n | n > 0\})$ is isomorphic to the infinite cyclic semigroup.

Fact 5.1. (i) *An identity is balanced if and only if it is satisfied by $\text{var}\{xy \approx yx\} = \text{var}S(\{a^n | n > 0\})$.*

(ii) *An identity is regular if and only if it is satisfied by $\text{var}\{x \approx xx, xy \approx yx\} = \text{var}S(\{\epsilon\})$.*

(iii) *For each $n > 0$, an identity is a \mathcal{P}_n -identity if and only if it is satisfied by $\text{var}\{t_1xt_2xt_3x \dots t_{n+1}x \approx x^{n+1}t_1t_2 \dots t_{n+1}, x^{n+1} \approx x^{n+2}\}^\delta = \text{var}S(W_n)$.*

In particular, an identity is a \mathcal{P}_1 -identity if and only if it is satisfied by $\text{var}\{x^2t \approx tx^2 \approx txx, x^2 \approx x^3\} = \text{var}S(\{ab\})$.

(iv) *An identity is block-balanced if and only if it is satisfied by $\text{var}\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta = \text{var}S(W_{AL})$.*

Proof. Parts (i) and (ii) are well-known.

(iii) The equality $\text{var}\{t_1xt_2xt_3x \dots t_{n+1}x \approx x^{n+1}t_1t_2 \dots t_{n+1}, x^{n+1} \approx x^{n+2}\}^\delta = \text{var}(S(W_n))$ is mentioned in [2]. The rest can be easily verified.

(iv) If an identity $\mathbf{u} \approx \mathbf{v}$ is block-balanced, then it is balanced, the order of linear letters in \mathbf{u} and \mathbf{v} is the same and the corresponding blocks of \mathbf{u} are permutations of the corresponding blocks of \mathbf{v} . Then by using the identities $\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta$ and commuting the adjacent occurrences of non-linear variables, it is easy to transform \mathbf{u} into \mathbf{v} .

If an identity $\mathbf{u} \approx \mathbf{v}$ is a consequence of $\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta$ then it is block-balanced because each identity in $\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta$ is block-balanced and the property of being a block-balanced identity is derivation-stable.

The equality $\text{var}\{\sigma_1, \sigma_\mu, \sigma_2\}^\delta = \text{var}S(W_{AL})$ holds by Corollary 3.7. \square

The main goal of this section is to prove Theorem 5.6 which implies Fact 5.1(iv) and contains six more statements similar to Fact 5.1(iv).

Definition 5.2. We say that a balanced identity $\mathbf{u} \approx \mathbf{v}$ satisfies

- (i) Property $\mathcal{P}_{1,1}$ if for each $x \neq y \in \text{Cont}(\mathbf{u})$ the pair $\{_{1\mathbf{u}x, 1\mathbf{u}y}\}$ is stable in $\mathbf{u} \approx \mathbf{v}$ (the order of first occurrences of variables is the same in \mathbf{u} and in \mathbf{v});
- (ii) Property $\mathcal{P}_{\ell,\ell}$ if for each $x \neq y \in \text{Cont}(\mathbf{u})$ the pair $\{_{\ell\mathbf{u}x, \ell\mathbf{u}y}\}$ is stable in $\mathbf{u} \approx \mathbf{v}$ (the order of last occurrences of variables is the same in \mathbf{u} and in \mathbf{v});
- (iii) Property $\mathcal{P}_{1,\ell}$ if for each $x \neq y \in \text{Cont}(\mathbf{u})$ the pair $\{_{1\mathbf{u}x, \ell\mathbf{u}y}\}$ is stable in $\mathbf{u} \approx \mathbf{v}$;
- (iv) Property $\mathcal{P}_{1,i}$ if for each $x \neq y \in \text{Cont}(\mathbf{u})$ and each $1 \leq i \leq \text{occ}_{\mathbf{u}}(y)$ the pair $\{_{1\mathbf{u}x, i\mathbf{u}y}\}$ is stable in $\mathbf{u} \approx \mathbf{v}$;
- (v) Property $\mathcal{P}_{i,\ell}$ if for each $x \neq y \in \text{Cont}(\mathbf{u})$ and each $1 \leq i \leq \text{occ}_{\mathbf{u}}(x)$ the pair $\{_{i\mathbf{u}x, \ell\mathbf{u}y}\}$ is stable in $\mathbf{u} \approx \mathbf{v}$.

We say that a set of identities Σ is *full* if each identity $(\mathbf{u} \approx \mathbf{v}) \in \Sigma$ satisfies the following condition:

(*) If the words \mathbf{u} and \mathbf{v} do not begin (end) with the same linear letter, then Σ contains the identity $t\mathbf{u} \approx t\mathbf{v}$ ($\mathbf{u}t \approx \mathbf{v}t$) for some $t \notin \text{Cont}(\mathbf{u}\mathbf{v})$.

For example, if Σ is a full set of identities containing $\sigma_\mu: xt_1xyt_2y \approx xt_1yxt_2y$, then Σ must also contain the identities $txt_1xyt_2y \approx txt_1yxt_2y$, $xt_1xyt_2yt \approx xt_1yxt_2yt$ and $txt_1xyt_2yt_3 \approx txt_1yxt_2yt_3$.

A *substitution* $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ is a homomorphism of the free semigroup \mathfrak{A}^+ . Let Σ be a full set of identities. A *derivation* of an identity $\mathbf{U} \approx \mathbf{V}$ from Σ is a sequence of words $\mathbf{U} = \mathbf{U}_1 \approx \mathbf{U}_2 \approx \dots \approx \mathbf{U}_l = \mathbf{V}$ and substitutions $\Theta_1, \dots, \Theta_{l-1}(\mathfrak{A} \rightarrow \mathfrak{A}^+)$ so that for each $i = 1, \dots, l-1$ we have $\mathbf{U}_i = \Theta_i(\mathbf{u}_i)$ and $\mathbf{U}_{i+1} = \Theta_i(\mathbf{v}_i)$ for some identity $\mathbf{u}_i \approx \mathbf{v}_i \in \Sigma$. It is easy to see that each finite set of identities Σ is a subset of a finite full set of identities Σ' so that $\text{var}\Sigma = \text{var}\Sigma'$ and that an identity τ can be derived from Σ in the usual sense if and only if τ can be derived from Σ' in the sense defined in the previous sentence.

We say that a property \mathcal{P} of identities is *substitution-stable* provided that for every substitution $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$, the identity $\Theta(\mathbf{u}) \approx \Theta(\mathbf{v})$ satisfies property \mathcal{P} whenever $\mathbf{u} \approx \mathbf{v}$ satisfies \mathcal{P} . Evidently, a property of identities is *derivation-stable* if and only if it is transitive and substitution-stable.

Let $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution so that $\Theta(\mathbf{u}) = \mathbf{U}$. Then Θ induces a map $\Theta_{\mathbf{u}}$ from $\text{OccSet}(\mathbf{u})$ into subsets of $\text{OccSet}(\mathbf{U})$ as follows. If $1 \leq i \leq \text{occ}_{\mathbf{u}}(x)$ then $\Theta_{\mathbf{u}}(i\mathbf{u}x)$ denotes the set of all elements of $\text{OccSet}(\mathbf{U})$ contained in the subword of \mathbf{U} of the form $\Theta(x)$ that corresponds to the i^{th} occurrence of variable x in \mathbf{u} . For example, if $\Theta(x) = ab$ and $\Theta(y) = bab$ then $\Theta_{xyx}(2(xy)x) = \{_{3(abbabab)a, 4(abbabab)b}\}$. Evidently, for each $x \in \text{OccSet}(\mathbf{u})$ the set $\Theta_{\mathbf{u}}(x)$ is an interval in $(\text{OccSet}(\mathbf{U}), <_{\mathbf{U}})$. Now we define a function $\Theta_{\mathbf{u}}^{-1}$ from $\text{OccSet}(\mathbf{U})$ to $\text{OccSet}(\mathbf{u})$ as follows. If $c \in \text{OccSet}(\mathbf{U})$ then $\Theta_{\mathbf{u}}^{-1}(c) = d$ so that $\Theta_{\mathbf{u}}(d)$ contains c . For example, $\Theta_{xyx}^{-1}(3(abbabab)a) = 2(xy)x$. It is easy to see that if $\mathbf{U} = \Theta(\mathbf{u})$ then function $\Theta_{\mathbf{u}}^{-1}$ is a homomorphism from $(\text{OccSet}(\mathbf{U}), <_{\mathbf{U}})$ to $(\text{OccSet}(\mathbf{u}), <_{\mathbf{u}})$, i.e. for every $c, d \in \text{OccSet}(\mathbf{U})$ we have

$\Theta_{\mathbf{u}}^{-1}(c) \leq_{\mathbf{u}} \Theta_{\mathbf{u}}^{-1}(d)$ whenever $c <_{\mathbf{U}} d$. The following lemma is needed only to prove Theorem 5.4.

Lemma 5.3. *Let $\mathbf{u} \approx \mathbf{v}$ be a $\mathcal{P}_{1,1}$ -identity and $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution. If $\mathbf{U} = \Theta(\mathbf{u})$ and $\mathbf{V} = \Theta(\mathbf{v})$ then for each $x \in \text{Cont}(\mathbf{U})$ we have $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}x) = {}_{1\mathbf{u}}z$ and $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}x) = {}_{1\mathbf{v}}z$ for some $z \in \text{Cont}(\mathbf{u})$.*

Proof. Evidently, $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}x) = {}_{1\mathbf{u}}z$ and $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}x) = {}_{1\mathbf{v}}y$ for some $z, y \in \text{Cont}(\mathbf{u})$. If $z \neq y$ then both $\Theta(z)$ and $\Theta(y)$ contain x . Therefore, ${}_{1\mathbf{u}}z <_{\mathbf{u}} {}_{1\mathbf{u}}y$ and ${}_{1\mathbf{v}}y <_{\mathbf{v}} {}_{1\mathbf{v}}z$. To avoid a contradiction to the fact that the set $\{{}_{1\mathbf{u}}z, {}_{1\mathbf{u}}y\} \subseteq \text{OccSet}(\mathbf{u})$ is stable in $\mathbf{u} \approx \mathbf{v}$, we must assume that $y = z$. \square

Theorem 5.4. *All properties of identities in Definition 5.2 are derivation-stable.*

Proof. (i) Let $\mathbf{u} \approx \mathbf{v}$ be a $\mathcal{P}_{1,1}$ -identity and $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution. Denote $\mathbf{U} = \Theta(\mathbf{u})$ and $\mathbf{V} = \Theta(\mathbf{v})$. Suppose that for some $x, y \in \text{Cont}(\mathbf{U})$ we have ${}_{1\mathbf{U}}x <_{\mathbf{U}} {}_{1\mathbf{U}}y$. Then by Lemma 5.3 we have $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}x) = {}_{1\mathbf{u}}z$, $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}x) = {}_{1\mathbf{v}}z$ for some $z \in \text{Cont}(\mathbf{u})$, $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}y) = {}_{1\mathbf{u}}p$ and $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}y) = {}_{1\mathbf{v}}p$ for some $p \in \text{Cont}(\mathbf{u})$.

Since $\Theta_{\mathbf{u}}^{-1}$ is a homomorphism from $(\text{OccSet}(\mathbf{U}), <_{\mathbf{U}})$ to $(\text{OccSet}(\mathbf{u}), <_{\mathbf{u}})$, we have that ${}_{1\mathbf{u}}z \leq_{\mathbf{u}} {}_{1\mathbf{u}}p$. Since the identity $\mathbf{u} \approx \mathbf{v}$ satisfies Property $\mathcal{P}_{1,1}$, we have ${}_{1\mathbf{v}}z \leq_{\mathbf{v}} {}_{1\mathbf{v}}p$. If $z \neq p$ then we have ${}_{1\mathbf{v}}x <_{\mathbf{v}} {}_{1\mathbf{v}}y$ because the map $l_{\mathbf{U}, \mathbf{V}}$ restricted to the set $\{{}_{1\mathbf{U}}x, {}_{1\mathbf{U}}y\}$ is a composition of three isomorphisms: $\Theta_{\mathbf{u}}^{-1} \circ l_{\mathbf{u}, \mathbf{v}} \circ (\Theta_{\mathbf{v}}^{-1})^{-1}$.

If $z = p$ then using the fact that the ordered sets $(\Theta_{\mathbf{u}}({}_{1\mathbf{u}}z), <_{\mathbf{U}})$ and $(\Theta_{\mathbf{v}}({}_{1\mathbf{v}}z), <_{\mathbf{V}})$ correspond to the same word $\Theta(z)$, it is easy to show that ${}_{1\mathbf{v}}x <_{\mathbf{v}} {}_{1\mathbf{v}}y$. In either case, the pair $\{{}_{1\mathbf{U}}x, {}_{1\mathbf{U}}y\}$ is left-stable in $\mathbf{U} \approx \mathbf{V}$. Therefore, the identity $\mathbf{U} \approx \mathbf{V}$ also satisfies Property $\mathcal{P}_{1,1}$. Thus, we have proved that Property $\mathcal{P}_{1,1}$ is substitution-stable.

(ii) Let $\mathbf{u} \approx \mathbf{v}$ be a $\mathcal{P}_{1,\ell}$ -identity and $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution. Denote $\mathbf{U} = \Theta(\mathbf{u})$ and $\mathbf{V} = \Theta(\mathbf{v})$. Suppose that for some $x, y \in \text{Cont}(\mathbf{U})$ we have ${}_{\ell\mathbf{U}}x <_{\mathbf{U}} {}_{1\mathbf{U}}y$. Evidently, $\Theta_{\mathbf{u}}^{-1}({}_{\ell\mathbf{U}}x) = {}_{\ell\mathbf{u}}x$ and $\Theta_{\mathbf{v}}^{-1}({}_{\ell\mathbf{V}}x) = {}_{\ell\mathbf{v}}x'$ for some $x, x' \in \text{Cont}(\mathbf{u})$. Also, $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}y) = {}_{1\mathbf{u}}y$ and $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}y) = {}_{1\mathbf{v}}y'$ for some $y, y' \in \text{Cont}(\mathbf{u})$.

Since $\Theta_{\mathbf{u}}^{-1}$ is a homomorphism from $(\text{OccSet}(\mathbf{U}), <_{\mathbf{U}})$ to $(\text{OccSet}(\mathbf{u}), <_{\mathbf{u}})$, we have that ${}_{\ell\mathbf{u}}x \leq_{\mathbf{u}} {}_{1\mathbf{u}}y$. Since both $\Theta(x)$ and $\Theta(x')$ contain x and both $\Theta(y)$ and $\Theta(y')$ contain y , we have ${}_{\ell\mathbf{u}}x' \leq_{\mathbf{u}} {}_{\ell\mathbf{u}}x \leq_{\mathbf{u}} {}_{1\mathbf{u}}y \leq_{\mathbf{u}} {}_{1\mathbf{u}}y'$. Since the identity $\mathbf{u} \approx \mathbf{v}$ satisfies Property $\mathcal{P}_{1,\ell}$, we have ${}_{\ell\mathbf{v}}x' \leq_{\mathbf{v}} {}_{1\mathbf{v}}y'$.

If $x' \neq y'$ then we have ${}_{\ell\mathbf{v}}x <_{\mathbf{v}} {}_{1\mathbf{v}}y$ because the map $l_{\mathbf{U}, \mathbf{V}}$ restricted to the set $\{{}_{1\mathbf{U}}x, {}_{\ell\mathbf{U}}y\}$ is a composition of three isomorphisms: $\Theta_{\mathbf{u}}^{-1} \circ l_{\mathbf{u}, \mathbf{v}} \circ (\Theta_{\mathbf{v}}^{-1})^{-1}$. If $x' = y'$ then using the fact that the ordered sets $(\Theta_{\mathbf{u}}({}_{1\mathbf{u}}x'), <_{\mathbf{U}})$ and $(\Theta_{\mathbf{v}}({}_{1\mathbf{v}}x'), <_{\mathbf{V}})$ correspond to the same word $\Theta(x')$, it is easy to show that ${}_{\ell\mathbf{v}}x <_{\mathbf{v}} {}_{1\mathbf{v}}y$. In either case, the pair $\{{}_{\ell\mathbf{U}}x, {}_{1\mathbf{U}}y\}$ is left-stable in $\mathbf{U} \approx \mathbf{V}$. Therefore, the identity $\mathbf{U} \approx \mathbf{V}$ also satisfies Property $\mathcal{P}_{1,\ell}$. Thus, we have proved that Property $\mathcal{P}_{1,\ell}$ is substitution-stable.

(iii) Let $\mathbf{u} \approx \mathbf{v}$ be a $\mathcal{P}_{1,i}$ -identity and $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^+$ be a substitution. Denote $\mathbf{U} = \Theta(\mathbf{u})$ and $\mathbf{V} = \Theta(\mathbf{v})$. Let $x \neq y \in \text{Cont}(\mathbf{U})$. Since Property $\mathcal{P}_{1,i}$ is stronger than $\mathcal{P}_{1,1}$, by Lemma 5.3 we may assume that $\Theta_{\mathbf{u}}^{-1}({}_{1\mathbf{U}}x) = {}_{1\mathbf{u}}x$ and $\Theta_{\mathbf{v}}^{-1}({}_{1\mathbf{V}}x) =$

${}_{1\mathbf{v}}x$. Since $\mathbf{u} \approx \mathbf{v}$ is a balanced identity we identify $\text{OccSet}(\mathbf{u})$ and $\text{OccSet}(\mathbf{v})$. In particular, we identify ${}_{1\mathbf{u}}x$ and ${}_{1\mathbf{v}}x$.

Define $\Theta_{\mathbf{u}}^{-1}(y) := \{c \in \text{OccSet}(\mathbf{u}) \mid c = \Theta_{\mathbf{u}}^{-1}({}_i\mathbf{u}y), 1 \leq i \leq \text{occ}_{\mathbf{u}}(y)\}$. Define $Y_{\mathbf{u}} := \{c \in \Theta_{\mathbf{u}}^{-1}(y) \mid c \leq_{\mathbf{u}} ({}_{1\mathbf{u}}x)\}$. Since $\mathbf{u} \approx \mathbf{v}$ satisfies Property $\mathcal{P}_{1,i}$, we have $Y_{\mathbf{u}} = Y_{\mathbf{v}}$. This implies that the number of occurrences of y which precede ${}_{1\mathbf{u}}x$ in \mathbf{U} is the same as the number of occurrences of y which precede ${}_{1\mathbf{v}}x$ in \mathbf{V} . Therefore, the identity $\mathbf{U} \approx \mathbf{V}$ also satisfies Property $\mathcal{P}_{1,i}$.

Thus, we have proved that Property $\mathcal{P}_{1,1}$ is substitution-stable. Properties $\mathcal{P}_{\ell,\ell}$ and $\mathcal{P}_{i,\ell}$ are substitution-stable by dual arguments. Since all properties of identities in Definition 5.2 are transitive (obvious) and substitution-stable, all these properties are derivation-stable. \square

With each subset Σ of $\{\sigma_1, \sigma_{\mu}, \sigma_2\}$ we associate an assignment of two Types to all pairs of occurrences of distinct non-linear variables in all words as follows. We say that each pair of occurrences of two distinct non-linear variables in each word is $\{\sigma_1, \sigma_{\mu}, \sigma_2\}$ -good. If Σ is a proper subset of $\{\sigma_1, \sigma_{\mu}, \sigma_2\}$, then we say that a pair of occurrences of distinct non-linear variables is Σ -good if it is not declared to be Σ -bad in the following definition.

Definition 5.5. *If $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ is a pair of occurrences of two distinct non-linear variables $x \neq y$ in a word \mathbf{u} then*

- (i) *pair $\{c, d\}$ is $\{\sigma_{\mu}, \sigma_2\}$ -bad if $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}y\}$;*
- (ii) *pair $\{c, d\}$ is $\{\sigma_1, \sigma_{\mu}\}$ -bad if $\{c, d\} = \{{}_{\ell\mathbf{u}}x, {}_{\ell\mathbf{u}}y\}$;*
- (iii) *pair $\{c, d\}$ is $\{\sigma_1, \sigma_2\}$ -bad if $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{\ell\mathbf{u}}y\}$.*
- (iv) *pair $\{c, d\}$ is σ_{μ} -bad if $\{c, d\} = \{{}_{1\mathbf{u}}x, {}_{1\mathbf{u}}y\}$ or $\{c, d\} = \{{}_{\ell\mathbf{u}}x, {}_{\ell\mathbf{u}}y\}$;*
- (v) *pair $\{c, d\}$ is σ_{ℓ} -bad if $c = {}_{1\mathbf{u}}x$ or $d = {}_{1\mathbf{u}}y$;*
- (vi) *pair $\{c, d\}$ is σ_1 -bad if $c = {}_{\ell\mathbf{u}}x$ or $d = {}_{\ell\mathbf{u}}y$.*

The following theorem describes the equational theories for each of the seven varieties defined by the seven subsets of $\{\sigma_1, \sigma_{\mu}, \sigma_2\}$. It also implies Fact 5.1(iv).

Theorem 5.6. *If $\Sigma \subseteq \{\sigma_1, \sigma_{\mu}, \sigma_2\}$ then for every identity $\mathbf{u} \approx \mathbf{v}$ the following conditions are equivalent:*

- (i) *$\mathbf{u} \approx \mathbf{v}$ is block-balanced and each Σ -bad pair of occurrences of two distinct non-linear variables in \mathbf{u} is stable in $\mathbf{u} \approx \mathbf{v}$;*
- (ii) *$\mathbf{u} \approx \mathbf{v}$ can be derived from Σ^{δ} by swapping Σ -good adjacent pairs of occurrences;*
- (iii) *$\mathbf{u} \approx \mathbf{v}$ is satisfied by $\text{var}(\Sigma^{\delta})$.*

Proof. (i) \rightarrow (ii) We assign a Type to each pair $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ of occurrences of distinct variables in a word \mathbf{u} as follows. If $\{c, d\}$ is Σ -good then we say that $\{c, d\}$ is of Type 1. Otherwise, $\{c, d\}$ is of Type 2.

Let $\mathbf{u} \approx \mathbf{v}$ be a block-balanced identity so that each Σ -bad pair of occurrences of two distinct non-linear variables in \mathbf{u} is stable in $\mathbf{u} \approx \mathbf{v}$. Let $\{c, d\} \subseteq \text{OccSet}(\mathbf{u})$ be a critical pair in $\mathbf{u} \approx \mathbf{v}$. Suppose that $\{c, d\}$ is of Type 1. Then using an identity from Σ^{δ} and swapping c and d in \mathbf{u} we obtain some word \mathbf{w} . Evidently, the identity

$\mathbf{w} \approx \mathbf{v}$ satisfies both properties required by Lemma 3.2. Notice that the identity $\mathbf{u} \approx \mathbf{v}$ does not have any unstable pairs of Type 2.

(ii) \rightarrow (iii) Obvious.

(iii) \rightarrow (i) Notice that each identity in $(\mathbf{u} \approx \mathbf{v}) \in \Sigma^\delta$ is block-balanced and each Σ -bad pair of occurrences of two distinct non-linear variables in \mathbf{u} is stable in $\mathbf{u} \approx \mathbf{v}$. By Fact 5.1(iv) and Theorem 5.4 this property is derivation-stable. \square

Here are notation-free reformulations of some statements contained in Theorem 5.6.

Corollary 5.7. (i) *An identity is a consequence of $\{\sigma_\mu\}^\delta$ if and only if it is block-balanced and the orders of the first and the last occurrences of variables in its left and right sides are the same;*

(ii) *An identity is a consequence of $\{\sigma_1, \sigma_\mu\}^\delta$ if and only if it is block-balanced and the order of the last occurrences of variables in its left and right sides is the same;*

(iii) *An identity is a consequence of $\{\sigma_2, \sigma_\mu\}^\delta$ if and only if it is block-balanced and the order of the first occurrences of variables in its left and right sides is the same.*

Given a set of identities Σ we say that a word \mathbf{u} is a Σ -word if $S(\{\mathbf{u}\}) \models \Sigma$. For example, the word $at_1bbbt_2cct_3aa$ is a $\{\sigma_\mu, \sigma_1, \sigma_2\}$ -word. In view of the result of Jackson ([1, Lemma 3.3]), a word \mathbf{u} is a Σ -word if and only if \mathbf{u} is an isoterm for $\text{var}\Sigma$. It is shown in [2] that if W is a set of words then $S(W)$ is equationally equivalent to the direct product of $S(\{\mathbf{u}\})$ for all $\mathbf{u} \in W$. This implies that $S(W) \models \Sigma$ if and only if each word in W is a Σ -word. The result of Lee (Theorem 3.5 above) immediately implies the following.

Corollary 5.8. *If W is a set of $\{\sigma_1, \sigma_\mu\}$ -words or W is a set of $\{\sigma_\mu, \sigma_2\}$ -words then the monoid $S(W)$ is finitely based.*

Evidently, every almost-linear word is a $\{\sigma_1, \sigma_\mu, \sigma_2\}$ -word and consequently, it is a $\{\sigma_1, \sigma_\mu\}$ -word and a $\{\sigma_\mu, \sigma_2\}$ -word. So, Corollary 5.8 generalizes Theorem 3.2 in [9] that says that every set of almost-linear words is finitely based. The next statement together with Definition 5.5 gives us a simple algorithm that recognizes Σ -words in the seven varieties that we are interested in.

Lemma 5.9. *For each $\Sigma \subseteq \{\sigma_1, \sigma_\mu, \sigma_2\}$, a word \mathbf{U} is a Σ -word if and only if every adjacent pair of occurrences of two distinct non-linear variables in \mathbf{U} is Σ -bad.*

Proof. \Rightarrow Suppose that every adjacent pair of occurrences of two distinct non-linear variables in \mathbf{U} is Σ -bad. If $(\mathbf{u} \approx \mathbf{v}) \in \Sigma$ then $\mathbf{u} \approx \mathbf{v}$ is a block-balanced identity with two non-linear variables x and y . Let $\Theta : \mathfrak{A} \rightarrow \mathfrak{A}^*$ be a substitution. If $\Theta(x)\Theta(y) = \Theta(y)\Theta(x)$ then $\Theta(\mathbf{u}) = \Theta(\mathbf{v})$. If $\Theta(x)\Theta(y) \neq \Theta(y)\Theta(x)$ then it is easy to see that both $\Theta(\mathbf{u})$ and $\Theta(\mathbf{v})$ contain Σ -good adjacent pairs of occurrences and consequently, $S(\{\mathbf{U}\})$ satisfies $\mathbf{u} \approx \mathbf{v}$.

\Leftarrow Now suppose that \mathbf{U} is a Σ -word, that is $S(\{\mathbf{U}\}) \models \Sigma$. To obtain a contradiction, let us assume that \mathbf{U} contains a Σ -good adjacent pair of occurrences of two

distinct non-linear variables $\{c, d\} \subseteq \text{OccSet}(\mathbf{U})$. Then one of the identities in Σ is applicable to \mathbf{U} . Therefore, $S(\{\mathbf{U}\}) \models \mathbf{U} \approx \mathbf{V}$ so that the word \mathbf{V} is obtained from \mathbf{U} by swapping c and d . This contradicts the fact that \mathbf{U} is an isoterms for $S(\{\mathbf{U}\})$. So, we must assume that every adjacent pair of occurrences of two distinct non-linear variables in \mathbf{U} is Σ -bad. \square

Lemma 5.9 will be refined and used in [11, 12].

Corollary 5.10. *A word \mathbf{U} is a $\{\sigma_\mu, \sigma_1, \sigma_2\}$ -word if and only if each block in \mathbf{U} depends on at most one variable.*

In view of Corollary 5.10, we will refer to $\{\sigma_\mu, \sigma_1, \sigma_2\}$ -words as to *block-1-simple* words. It will be verified in [12] that if \mathbf{U} is a block-1-simple word then $\text{var}S(\{\mathbf{U}\}) = \text{var}S(W)$ for some set of almost-linear words W .

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